Trigonometry

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COURSE

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Mathematics, Trigonometry

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Although the words "he," "him," and "his" are used sparingly in this course to enhance communication, they are not intended to be gender driven or to affront or discriminate against anyone.

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CHAPTER 3

TRIGONOMETRIC MEASUREMENTS

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Measure angles in degrees, radians, and mils.
2. Find angular velocity and the area of a sector using radians.
3. Apply the Pythagorean theorem and properties of similar right triangles to problem solving.
4. Apply trigonometric ratios, functions, and tables to problem solving.

INTRODUCTION

This is the first of several chapters in this course dealing with the subject of trigonometry. Chapters 4, 5, and 6 also deal directly with triangles and trigonometry. Chapter 7 deals with vectors and forces. The study of vectors and forces is so closely related to trigonometry that it is normally included in a trigonometry course.

Mathematics, Volume 1, introduces numerical trigonometry and some applications in problem solving. However, trigonometry is not restricted to solving problems involving triangles; it also forms a foundation for some advanced mathematical concepts and subject areas. Trigonometry is both algebraic and geometric in nature, and in this course both of these qualities will be applied.

MEASURING ANGLES

Mathematics, Volume 1, pointed out that an angle is formed when two straight lines intersect. In this course, an angle is considered to be generated when a line having a set direction is rotated about a point, as depicted in figure 3-1.

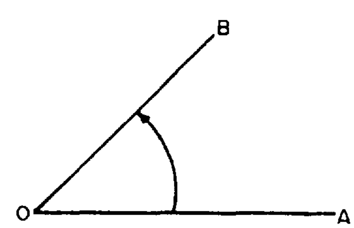


Figure 3-1.—Generation of an angle.

In figure 3-1, line *OA* is laid out as a reference line having a set direction. One end of the line is used as a pivot point and the line is rotated from its initial position (line *OA*) to another position (line *OB*), as in opening a door. As the line turns on its pivot point, it generates the angle *AOB*.

The following terminology is used in this and subsequent chapters:

1. *Radius vector*—The line that is rotated to generate an angle.
2. *Initial position*—The original position of the radius vector; corresponds to line *OA* in figure 3-1.
3. *Terminal position*—The final position of the radius vector; corresponds to line *OB* in figure 3-1.
4. *Positive angle*—The angle generated by rotating the radius vector counterclockwise from the initial position.
5. *Negative angle*—The angle generated by rotating the radius vector clockwise from the initial position.

The convention of identifying angles by use of Greek letters is followed in this text. When only one angle is involved, it will be symbolized by θ (theta). Other Greek letters will be used when more than one angle is involved. The additional symbols used will be ϕ (phi), α (alpha), and β (beta).

DEGREES

The degree system is the most common system of angular measurement. In this system a complete revolution is divided into 360 equal parts called *degrees*; so,

$$1 \text{ revolution} = 360^\circ$$

For accuracy, each degree is divided into 60 minutes; so,

$$1^\circ = 60'$$

Each minute is divided into 60 seconds; so,

$$1' = 60''$$

For convenience in working with angles, the 360° are divided into four equal parts of 90° each, similar to the rectangular coordinate system. The 90° sectors, called *quadrants*, are numbered according to the convention shown in figure 3-2.

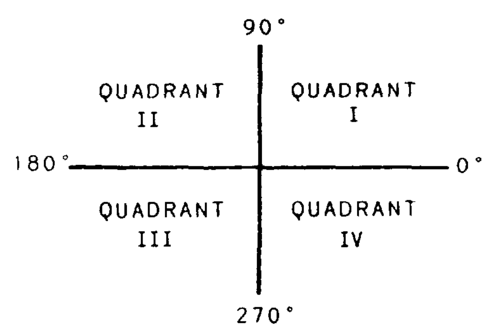


Figure 3-2.—Quadrant positions.

If the angle generated by rotating the radius vector in a positive (counterclockwise) direction is between 0° and 90°, then the angle is in the first quadrant. If the angle is between 90° and 180°, then the angle is in the second quadrant. If the angle is between 180° and 270°, then the angle is in the third quadrant. And if the angle is between 270° and 360°, then the angle is in the fourth quadrant.

If the angle generated by rotating the radius vector in a positive direction is more than 360°, then the quadrant in which the angle lies is found by subtracting from the angle the largest multiple of 360° that the angle contains. The quadrant in which the remainder angle lies is determined as described in the previous paragraph. *The original angle lies in the same quadrant as the remainder angle.*

EXAMPLE: In which quadrant is the angle 130°?

SOLUTION: Since 130° is between 90° and 180°, it is in the second quadrant. (See fig. 3-3, view A).

EXAMPLE: In which quadrant is the angle 850°?

SOLUTION: The largest multiple of 360° contained in 850° is 720°; so, 850° − 720° = 130°. Since 130° is in the second quadrant, then 850° also lies in the second quadrant. This relationship is shown in figure 3-3, view B.

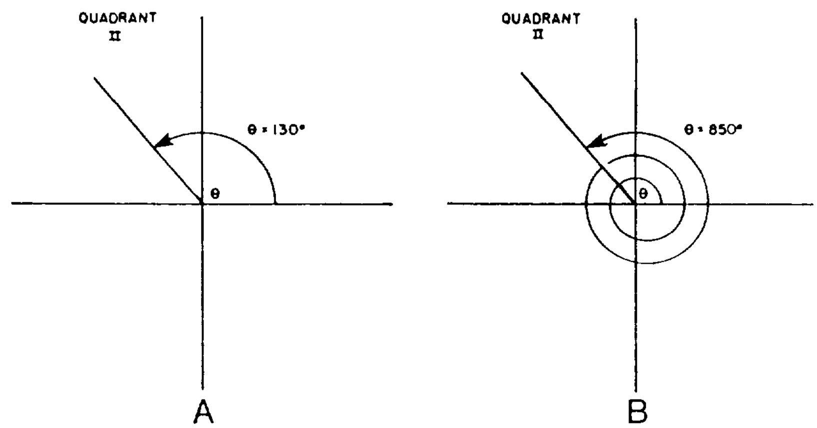


Figure 3-3.—Angle generation.

PRACTICE PROBLEMS:

Determine the quadrant in which each of the following angles lies:

1. 260°

2. 290°

3. 800°

4. 1,930°

ANSWERS:

1. 3rd

2. 4th

3. 1st

4. 2nd

RADIANS

Another even more fundamental method of angular measurement involves the *radian*. It has certain advantages over the degree method. Radian measurement greatly simplifies work with trigonometric functions in calculus. Radian measurement also relates the length of arc generated to the size of an angle.

A *radian* is defined as an angle that, if its vertex is placed at the center of a circle, intercepts an arc equal in length to the radius vector of the circle. Assume that an angle is generated, as shown in figure 3-4, view A. If we impose the condition that the length of the arc, *s*, described by the extremity of the line segment generating the angle, must

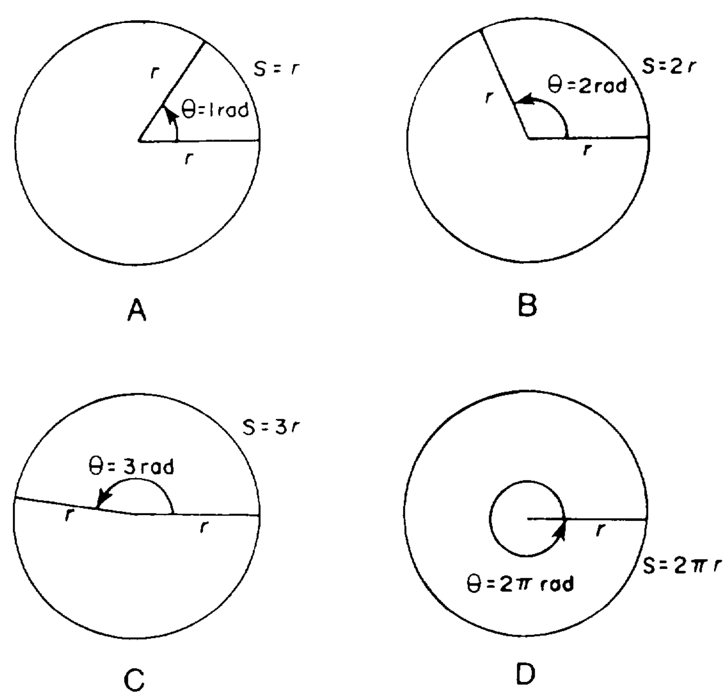


Figure 3-4.—Radian measure.

equal the length of the radius vector, r, then we would describe an angle exactly one radian in size; that is, for 1 radian,

$$s = r$$

In a broader sense, *the radian measure of an angle, θ, is the ratio of the length of the arc, s, it subtends to the length of the radius vector, r, of the circle in which it is the central angle*; that is,

$$\theta = \frac{s}{r}$$

For angle θ, in figure 3-4, view B, which intercepts an arc equal to two times the length of the radius vector, θ equals two radians. For angle θ, in figure 3-4, view C, which intercepts an arc equal to three times the length of the radius vector, θ equal three radians.

EXAMPLE: Find the radian measure of the central angle in a circle with a radius of 10 inches if the angle subtends an arc of 5 inches.

SOLUTION:

$$\theta = \frac{s}{r}$$

$$= \frac{5}{10}$$

$$= 0.5 \text{ radians}$$

Recall from plane geometry that the circumference of a circle is 2π times the radius or

$$C = 2\pi r$$

Hence, the radius vector can be laid off on the circumference 2π times. (See fig. 3-4, view D).

Since the arc length of the circumference is 2π radians and the circumference encompasses one complete revolution of 360°, then

$$2\pi \text{ radians} = 360°$$

One-half of a revolution equals 180° or π radians; so,

$$\pi \text{ radians} = 180° \qquad (3.1)$$

By dividing both sides of equation (3.1) by π, we find that

$$1 \text{ radian} = \frac{180^\circ}{\pi}$$

$$= 57.2958^\circ \text{ (rounded)}$$

$$= 57^\circ\ 17'\ 45''$$

By dividing both sides of equation (3.1) by 180, we find that

$$1^\circ = \frac{\pi}{180} \text{ radians}$$

$$= 0.01745 \text{ radians (rounded)}$$

NOTE: The degree symbol (°) is customarily used to indicate degrees, and a pure number with no symbol attached is used to indicate radians. For example, sin 3 should be understood to represent "sine of 3 radians," whereas the "sine of 3 degrees" would be written sin 3°.

The following list indicates other relationships frequently used in trigonometric problems:

Radians	Degrees
$\pi/6$	30
$\pi/4$	45
$\pi/3$	60
$\pi/2$	90
π	180
$3\pi/2$	270
2π	360

EXAMPLE: Express 160° in radians, using π in the answer.

SOLUTION:

$$1° = \frac{\pi}{180} \text{ radians}$$

$$160° = 160 \times 1°$$

$$= 160 \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{8\pi}{9} \text{ radians}$$

EXAMPLE: Express $\pi/20$ in degrees.

SOLUTION:

$$1 \text{ radian} = \frac{180°}{\pi}$$

$$\frac{\pi}{20} \text{ radians} = \frac{\pi}{20} \times 1 \text{ radian}$$

$$= \frac{\pi}{20} \times \frac{180°}{\pi}$$

$$= \frac{180°}{20}$$

$$= 9°$$

Refer to figure 3-5. We can see that if θ represents the number of radians in a central angle, r the length of the radius of the circle, and s the length of the intercepted arc, then the *length of the arc* equals the number of radians multiplied by the length of the radius or

$$s = \theta r$$

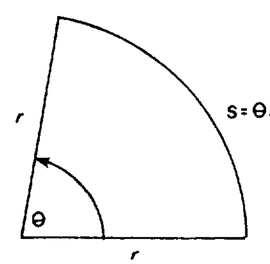


Figure 3-5.—Length of arc.

EXAMPLE: In a circle having a radius of 11 inches, an arc subtends a central angle of 3 radians. Find the length of the arc in inches.

SOLUTION:

$$s = \theta r$$

$$= 3 \cdot 11$$

$$= 33 \text{ inches}$$

PRACTICE PROBLEMS:

1. Find the number of radians in the central angle subtended by an arc 18 inches long in a circle whose radius is 8 inches.

Express the following angles in radians, using π in the answer:

2. $420°$

3. $135°$

Express the following angles in degrees:

4. 20π

5. $5\pi/6$

6. In a circle whose radius, r, is 4 inches, find in inches the length of arc, s, whose central angle is 1 1/4 radians.

ANSWER:

1. 9/4 radians

2. $7\pi/3$

3. $3\pi/4$

4. $3{,}600°$

5. $150°$

6. 5 inches

Because of the relationship of the radian to arc length, the radian has some special applications in measurements of angular velocity and area of a sector.

Angular Velocity

Another type of problem that radian measurement simplifies is that which relates the rotating motion of the wheels of a vehicle to its forward motion. Here we will not be dealing with angles alone but also with *angular velocity*. Let's analyze this type of motion.

Consider the circle at the left in figure 3-6 to indicate the original position of a wheel. As the wheel turns, it rolls so that the center moves along the line CC', where C' is the center of the wheel at its final position. The contact point at the bottom of the wheel moves an equal distance PP'; but as the wheel turns through angle θ, arc s is made to coincide with line PP'; so,

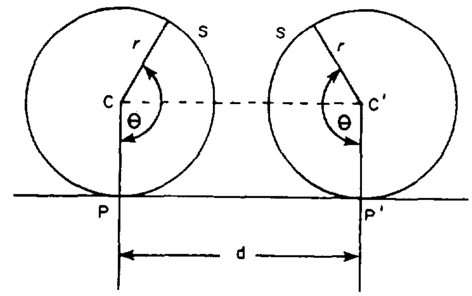


Figure 3-6.—Angular rotation.

$$s = PP' = d$$

or the length of arc is equal to the forward distance, d, the wheel travels. But since

$$s = r\theta$$

then the forward distance that the wheel travels is

$$d = r\theta$$

Dividing both sides of the previous equation by t gives

$$\frac{d}{t} = \frac{r\theta}{t}$$

When a vehicle moves with a constant velocity, v, in time, t, the distance, d, the vehicle travels is expressed by the formula

$$d = vt$$

Solving this formula for v, we have

$$v = \frac{d}{t}$$

The fraction d/t expresses the *linear velocity* of the vehicle, and θ/t is the *angular velocity*. If we let ω (Greek letter omega) stand for the angular velocity, then the equation

$$\frac{d}{t} = \frac{r\theta}{t}$$

becomes

$$v = r\omega$$

where ω is measured in radians per unit time.

EXAMPLE: A car wheel is rotating at 1,050 revolutions per minute (rpm). Find

1. the angular velocity in radians per second.
2. the linear velocity in meters per second on the tire tread, 25 centimeters from the center.

SOLUTION:

1. To find the angular velocity, we need to convert rev/min to rad/sec. To do this, we will apply unit conversions (multiples of one) as follows:

$$\omega = 1{,}050 \frac{\text{rev}}{\text{min}} \times 2\pi \frac{\text{rad}}{\text{rev}} \times \frac{1}{60} \frac{\text{min}}{\text{sec}}$$

$$= \frac{(1{,}050)(2\pi)}{60} \frac{\text{rad}}{\text{sec}}$$

$$= 35\pi \text{ radians per second}$$

2. We find the linear velocity as follows:

$$v = r\omega$$

$$= 25 \text{ cm} \times 35\pi \frac{\text{rad}}{\text{sec}}$$

$$= 875\pi \frac{\text{cm}}{\text{sec}}$$

NOTE: When no unit of angular measure is indicated, the angle is understood to be expressed in radians.

We now need to convert cm/sec to m/sec. We will again apply a unit conversion:

$$v = 875\pi \frac{\text{cm}}{\text{sec}} \times \frac{1}{100} \frac{\text{m}}{\text{cm}}$$

$$= \frac{875\pi}{100} \frac{\text{m}}{\text{sec}}$$

$$= 8.75\pi \text{ meters per second}$$

EXAMPLE: A car is traveling 40 miles per hour. If the wheel radius is 16 inches, what is the angular velocity of the wheels in

1. radians per minute?
2. revolutions per minute?

SOLUTION:

1. We know that

$$v = r\omega$$

Thus,

$$\omega = \frac{v}{r}$$

$$= \frac{40}{16}\ \frac{\text{mi/hr}}{\text{in}}$$

$$= \frac{5}{2}\ \frac{\text{mi}}{\text{hr} \times \text{in}} \times \frac{5{,}280}{1}\ \frac{\text{ft}}{\text{mi}} \times \frac{12}{1}\ \frac{\text{in}}{\text{ft}} \times \frac{1}{60}\ \frac{\text{hr}}{\text{min}}$$

$$= \frac{(5)(5{,}280)(12)}{(2)(60)}\ \frac{\text{rad}}{\text{min}}$$

$$= 2{,}640 \text{ radians per minute}$$

2. Since 2π radians $= 360°$ and $360° = 1$ revolution, then

$$\omega = 2{,}640\ \frac{\text{rad}}{\text{min}} \times \frac{1}{2\pi}\ \frac{\text{rev}}{\text{rad}}$$

$$= \frac{2{,}640}{2\pi}\ \frac{\text{rev}}{\text{min}}$$

$$= 420.2 \text{ revolutions per minute}$$

EXAMPLE: Determine the distance a truck will travel in 1 minute if the wheels are 3 feet in diameter and are turning at the rate of 5 revolutions per second. HINT: Diameter = 2 × radius

SOLUTION:

$$v = r\omega$$

$$\frac{d}{t} = r\omega$$

$$d = rt\omega$$

$$= \frac{3}{2} \text{ ft} \times 1 \text{ min} \times \left(5\ \frac{\text{rev}}{\text{sec}} \times \frac{2\pi}{1}\ \frac{\text{rad}}{\text{rev}}\right)$$

$$= \frac{3}{2} \text{ ft} \times 1 \text{ min} \times 10\pi\ \frac{\text{rad}}{\text{sec}} \times \frac{60}{1}\ \frac{\text{sec}}{\text{min}}$$

$$= \frac{(3)(10\pi)(60)}{2} \text{ ft}$$

$$= 2{,}827.43 \text{ feet}$$

Area of a Sector

From plane geometry we find that the area of the sector of a circle is proportional to the angle enclosed in the sector.

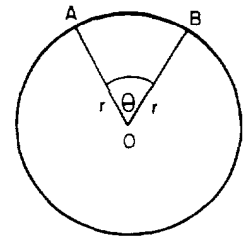


Figure 3-7.—Sector of a circle.

Consider sector AOB of the circle shown in figure 3-7. If θ is increased to 2π radians (360°), it encompasses the entire circle; so the area of the circle is proportional to 2π radians. Hence,

$$\frac{\text{area of sector}}{\text{area of circle}} = \frac{\theta}{2\pi}$$

But the area of a circle can be found by the formula

$$A = \pi r^2$$

By substitution, we find

$$\text{area of sector} = \frac{\theta}{2\pi}(\pi r^2)$$

$$= \frac{\theta r^2}{2}$$

Therefore, the *area of a sector of a circle* can be found by the formula

$$A = \frac{1}{2} r^2 \theta$$

where θ is expressed in radians.

EXAMPLE: Find the area of a sector of a circle with a radius of 6 inches having a central angle of 60°.

SOLUTION:

$$A = \frac{1}{2} r^2 \theta$$

$$= \frac{1}{2}(6 \text{ in})^2\left(60° \times \frac{\pi}{180°}\right)$$

$$= \frac{36\pi}{(2)(3)} \text{ in}^2$$

$$= 6\pi \text{ square inches}$$

The area of a sector of a circle can also be found if the radius and arc length are known. Since

$$s = r\theta$$

then

$$A = \frac{1}{2}r^2\theta$$

$$= \frac{1}{2}r(r\theta)$$

$$= \frac{1}{2}rs$$

EXAMPLE: What is the diameter of a circle if a sector of the circle has an arc length of 9 inches and an area of 18 square inches?

SOLUTION:

If

$$A = \frac{1}{2}rs$$

then

$$r = \frac{2A}{s}$$

$$= \frac{2(18 \text{ in}^2)}{9 \text{ in}}$$

$$= 4 \text{ in}$$

But

$$d = 2r$$

Therefore,

$$d = 2(4 \text{ in})$$

$$= 8 \text{ inches}$$

PRACTICE PROBLEMS:

1. A car travels 4,500 feet in 1 minute. The diameter of the wheels is 36 inches. What is the angular velocity of the wheels in radians per minute?

2. How far in feet does a car travel in 1 minute if the radius of the wheels is 18 inches and the angular velocity of the wheels is 1,000 radians per minute?

3. Find the area of a sector of a circle whose central angle is $\pi/3$ and whose diameter is 24 inches. Leave the answer in terms of π.

4. Find the area of a sector of a circle in inches whose arc length is 14 inches and whose radius is 2/3 feet.

ANSWERS:

1. 3,000 radians per minute

2. 1,500 feet

3. 24π square inches

4. 56 square inches

MILS

The *mil* is a unit of small angular measurement, which is not widely used but has some military applications in ranging and sighting. The *mil* is defined in two ways:

1. *As 1/6,400 of the circumference of a circle.*

2. *As the angle subtended by an object 1 unit long, perpendicular to the line of sight, at a distance of 1,000 units.*

From definition 1 we can see that since

$$360^\circ = 6{,}400 \text{ mils}$$

then

$$1^\circ = \frac{6{,}400}{360} \text{ mils}$$

$$= \frac{160}{9} \text{ mils}$$

$$= 17.78 \text{ mils (rounded)}$$

Also, since

$$6{,}400 \text{ mils} = 360^\circ$$

then

$$1 \text{ mil} = \frac{360^\circ}{6{,}400}$$

$$= \frac{9^\circ}{160}$$

$$= 0.05625^\circ$$

EXAMPLE: Convert 240 mils to degrees.

SOLUTION:

$$1 \text{ mil} = \frac{9^\circ}{160}$$

$$240 \text{ mils} = 240 \times 1 \text{ mil}$$

$$= 240 \times \frac{9^\circ}{160}$$

$$= \frac{27^\circ}{2}$$

$$= 13.5^\circ$$

EXAMPLE: Convert 27° to mils.

SOLUTION:

$$1° = \frac{160}{9} \text{ mils}$$

$$27° = 27 \times 1°$$

$$= 27 \times \frac{160}{9} \text{ mils}$$

$$= 480 \text{ mils}$$

Since

$$1 \text{ mil} = \frac{9°}{160}$$

and

$$1° = \frac{\pi}{180} \text{ radians}$$

then

$$1 \text{ mil} = \frac{9}{160} \times 1°$$

$$= \frac{9}{160} \times \frac{\pi}{180} \text{ radians}$$

$$= \frac{\pi}{3{,}200} \text{ radians}$$

$$= 0.00098 \text{ radians (rounded)}$$

We see that *1 mil is approximately 0.001* or *1/1,000 radians*. We also see that *1 radian ≈ 1,000 mils.*

EXAMPLE: Convert 25 mils to an approximate radian measure.

SOLUTION:

$$1 \text{ mil} \approx \frac{1}{1{,}000} \text{ radians}$$

$$25 \text{ mils} = 25 \times 1 \text{ mil}$$

$$\approx 25 \times \frac{1}{1{,}000} \text{ radians}$$

$$\approx \frac{25}{1{,}000} \text{ radians}$$

$$\approx 0.025 \text{ radians}$$

EXAMPLE: Convert 6.48 radians to an approximate measurement in mils.

SOLUTION:

$$1 \text{ radian} \approx 1{,}000 \text{ mils}$$

$$\begin{aligned} 6.48 \text{ radians} &= 6.48 \times 1 \text{ radian} \\ &\approx 6.48 \times 1{,}000 \text{ mils} \\ &\approx 6{,}480 \text{ mils} \end{aligned}$$

Referring to figure 3-8, when an angle, θ, subtended by an arc, s, is very small and the radius, r, is large, the chord, c, is almost equal to the arc, s.

The formula for the length of arc of a circle, as previously stated, is

$$s = r\theta$$

where θ is in radian measurement.

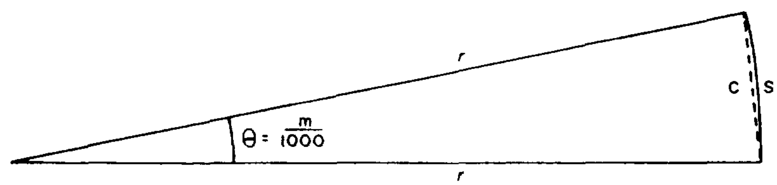


Figure 3-8.—Relationship of chord and arc.

If the measurement of the arc is made in mils, we must divide the mil measure by 1,000 to obtain the radian measure. Since,

$$1 \text{ mil} = \frac{1}{1{,}000} \text{ radians (approximately)}$$

then

$$m \text{ mils} = \frac{m}{1{,}000} \text{ radians}$$

So,

$$\begin{aligned} s &= r\left(\frac{m}{1{,}000}\right) \\ &= \frac{rm}{1{,}000} \end{aligned}$$

Now, since the chord, c, in figure 3-8, is approximately equal to the arc, s, then

$$c = \frac{rm}{1,000}$$

Now consider definition 2. If

$$r = 1,000 \text{ yds}$$

and

$$m = 1 \text{ mil}$$

then

$$c = \frac{rm}{1,000}$$

$$= \frac{1,000 \times 1}{1,000}$$

$$= 1 \text{ yard}$$

We also know that the arc, s, is approximately equal to 1 yard since $s \approx c$.

The military uses the fact that a mil subtends a yard at a distance of 1,000 yards for quick computations in the field.

EXAMPLE: Find the length of a target if, at a right angle to the line of sight, it subtends an angle of 15 mils at a range of 4,000 yards.

SOLUTION:

$$c = \frac{rm}{1,000}$$

$$= \frac{4,000 \times 15}{1,000}$$

$$= 60 \text{ yards}$$

EXAMPLE: A building known to be 80 feet long and perpendicular to the line of sight subtends an angle of 100 mils. What is the approximate range to the building?

SOLUTION:

Since

$$c = \frac{rm}{1{,}000}$$

then

$$r = \frac{1{,}000c}{m}$$

$$= \frac{1{,}000 \times 80}{100}$$

$$= 800 \text{ feet}$$

PRACTICE PROBLEMS:

1. Convert 3,456 mils to degrees.
2. Convert 12 degrees to mils.
3. Convert 27,183 mils to an approximate radian measure.
4. Convert 431 radians to an approximate measurement in mils.
5. A tower 500 feet away subtends a vertical angle of 250 mils. What is the height of the tower?
6. If points *X* and *Y* are 48 yards apart and are 4,000 yards from an observer, how many mils do they subtend?

ANSWERS:

1. 194.4°
2. 213.3 mils
3. 27.183 radians

4. 431,000 mils

5. 125 feet

6. 12 mils

PROPERTIES OF RIGHT TRIANGLES

Mathematics, Volume 1, contains information on the trigonometric ratios and other properties of triangles. This section restates some of the properties of right triangles for review and reference.

PYTHAGOREAN THEOREM

The *Pythagorean theorem* states that in a right triangle, the square of the length of the hypotenuse (longest side) is equal to the sum of the squares of the lengths of the other two sides. In the right triangle shown in figure 3-9, this relationship is expressed as

$$r^2 = x^2 + y^2$$

where r is the length of the hypotenuse and x and y are the lengths of the other two sides.

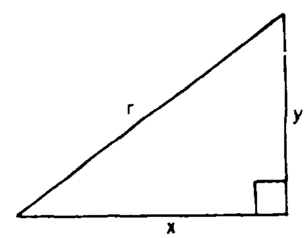


Figure 3-9.—Pythagorean relationship.

This relationship is useful in solving many problems and in developing trigonometric concepts.

EXAMPLE: In figure 3-10, what is the length of the hypotenuse of the right triangle if the lengths of the other two sides are 3 and 4?

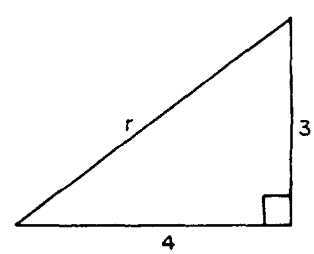


Figure 3-10.—Right triangle with hypotenuse unknown.

SOLUTION:

$$\begin{aligned} r^2 &= x^2 + y^2 \\ &= 4^2 + 3^2 \\ &= 16 + 9 \\ &= 25 \end{aligned}$$

So,

$$\begin{aligned} r &= \sqrt{25} \\ &= 5 \end{aligned}$$

NOTE: We will use the positive value of the square root since we are dealing with lengths.

EXAMPLE: Figure 3-11 shows a right triangle with a hypotenuse equal to 40 and one of the other sides equal to 10. What is the length of the remaining side?

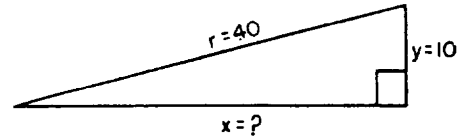


Figure 3-11.—Right triangle with one side unknown.

SOLUTION:

$$r^2 = x^2 + y^2$$

or

$$\begin{aligned} x^2 &= r^2 - y^2 \\ &= 40^2 - 10^2 \\ &= 1,600 - 100 \\ &= 1,500 \end{aligned}$$

So,

$$\begin{aligned} x &= \sqrt{1,500} \\ &= 38.7 \text{ (rounded)} \end{aligned}$$

SIMILAR RIGHT TRIANGLES

Another relationship of right triangles that is useful in trigonometry concerns *similar triangles*. Whenever the angles of one triangle are equal to the corresponding angles in another triangle, the two triangles are said to be *similar*.

For example, right triangle A in figure 3-12 is similar to right triangle B. Since the two triangles are similar by definition, the following proportions involving the lengths of the corresponding sides are true:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

This relationship can be used to find the lengths of unknown sides in similar triangles.

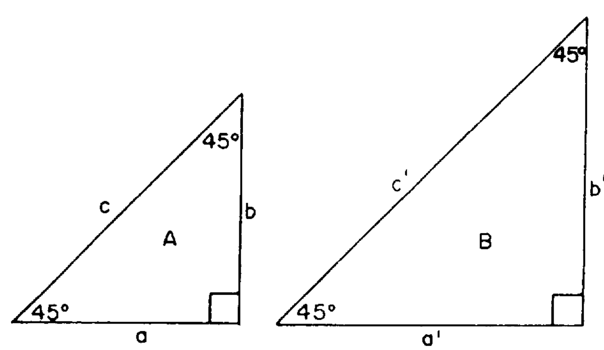


Figure 3-12.—Similar triangles.

EXAMPLE: Assume right triangles A and B in figure 3-13 are similar with lengths as shown. Find the lengths of sides b' and c'.

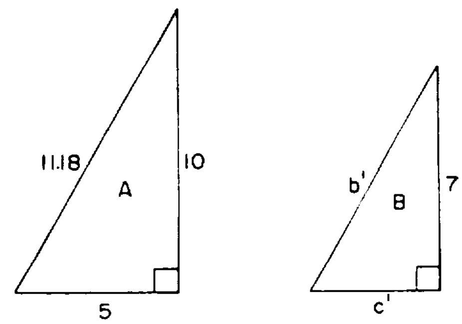


Figure 3-13.—Similar triangles, solution example.

SOLUTION:

Since

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

then

$$\frac{10}{7} = \frac{11.18}{b'} = \frac{5}{c'}$$

Side b' can be solved for using the first two ratios:

$$\frac{10}{7} = \frac{11.18}{b'}$$

So,

$$b' = \frac{11.18 \times 7}{10}$$

$$= \frac{78.26}{10}$$

$$= 7.826$$

Side c' can be solved for using the first and third ratios:

$$\frac{10}{7} = \frac{5}{c'}$$

So,

$$c' = \frac{5 \times 7}{10}$$

$$= 3.5$$

Recall from plane geometry that *the sum of the interior angles of any triangle is equal to 180°*. Using this fact, we can assume that two triangles are similar if two angles of one are equal to two angles of the other. The remaining angle in any triangle must be equal to 180° minus the sum of the other two angles.

If an acute angle of one right triangle is equal to an acute angle of another right triangle, the triangles are similar because the right angles in the two triangles are also equal to each other.

Hence, if θ is one of the acute angles in a right triangle, then $(90° - \theta)$ is the other acute angle, such that

$$90° + \theta + (90° - \theta) = 180°$$

Therefore, *two right triangles are similar if an acute angle of one triangle is equal to an acute angle of the other triangle.*

Many practical uses of trigonometry are based on the fact that two right triangles are similar if an acute angle of one triangle is equal to an acute angle of the other triangle.

In figure 3-14 triangle A is similar to triangle B since an acute angle in triangle A is equal to an acute angle in triangle B. Since triangle A is similar to triangle B, then

$$\frac{x}{x'} = \frac{y}{y'} = \frac{r}{r'}$$

Interchanging terms in the proportions gives

$$\frac{x}{y} = \frac{x'}{y'}$$

$$\frac{y}{r} = \frac{y'}{r'}$$

and

$$\frac{x}{r} = \frac{x'}{r'}$$

which are considered among the main principles of numerical trigonometry.

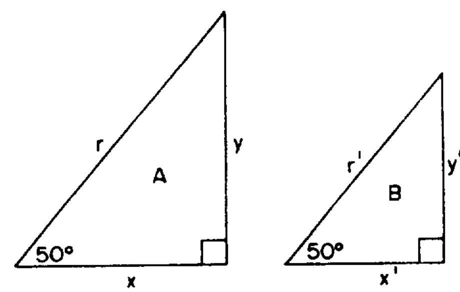


Figure 3-14.—Similar right triangles.

PRACTICE PROBLEMS:

Refer to figure 3-15 in solving the following problems:

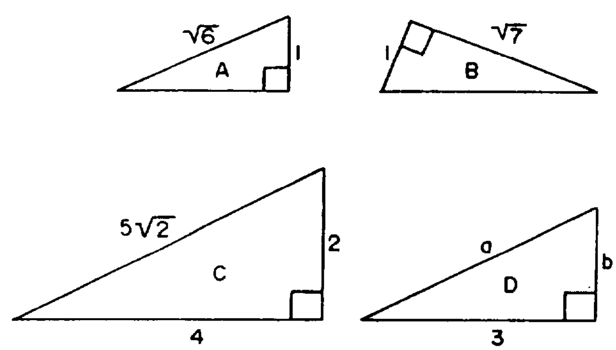


Figure 3-15.—Triangles for practice problems.

1. Use the Pythagorean theorem to calculate the unknown length in triangle A.

2. Use the Pythagorean theorem to calculate the unknown length in triangle *B*.

3. Triangles *C* and *D* are similar triangles. Find the length of sides *a* and *b* in triangle *D*.

ANSWERS:

1. $\sqrt{5}$

2. $2\sqrt{2}$

3. $a = 15\sqrt{2}/4$, $b = 3/2$

TRIGONOMETRIC RATIOS, FUNCTIONS, AND TABLES

The properties of triangles given in the previous section provide a means for solving many practical problems. Certain practical problems, however, require knowledge of right triangle relationships other than the Pythagorean theorem or the relationships of similar triangles before solutions can be found.

For example, the following two problems require additional knowledge:

1. Find the values of the unknown sides and angles in a right triangle when the values of one side and one acute angle are given.

2. Find the value of the unknown side and the values of the angles in a right triangle when two sides are known.

The additional relationships between the sides and angles of a right triangle are called *trigonometric ratios*. These ratios were introduced in *Mathematics*, Volume 1, and are reviewed in the following paragraphs. The basic foundations of trigonometry rest upon these ratios and their associated trigonometric functions.

TRIGONOMETRIC RATIOS AND FUNCTIONS

The sides of a right triangle form six ratios. In figure 3-16 we will use the acute angle θ and the three sides x, y, and r two at a time to define the trigonometric ratios. These ratios and the trigonometric functions associated with each ratio are listed as follows:

$$\text{the sine of } \theta = \frac{y}{r}, \text{ written } \sin \theta$$

$$\text{the cosine of } \theta = \frac{x}{r}, \text{ written } \cos \theta$$

$$\text{the tangent of } \theta = \frac{y}{x}, \text{ written } \tan \theta$$

$$\text{the cotangent of } \theta = \frac{x}{y}, \text{ written } \cot \theta$$

$$\text{the secant of } \theta = \frac{r}{x}, \text{ written } \sec \theta$$

$$\text{the cosecant of } \theta = \frac{r}{y}, \text{ written } \csc \theta$$

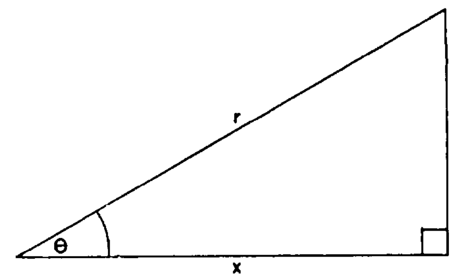


Figure 3-16.—Right triangle for determining ratios.

The trigonometric functions of a right triangle are remembered easier by the convention of naming the sides. Refer to figure 3-17. The side of length y is called the side *opposite* angle θ, the side of length x is called the side *adjacent* to angle θ, and the side of length r is called the *hypotenuse*. Using this terminology causes the six trigonometric functions to be defined as:

$$\sin \theta = \frac{y}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos \theta = \frac{x}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan \theta = \frac{y}{x} = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cot \theta = \frac{x}{y} = \frac{\text{adjacent}}{\text{opposite}}$$

$$\sec \theta = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\csc \theta = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{opposite}}$$

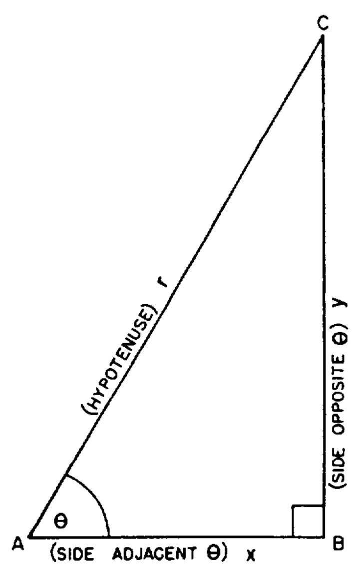


Figure 3-17.—Names of sides of a right triangle.

Remember that the six trigonometric ratios apply only to the acute angles of a right triangle.

EXAMPLE: Give the values of the trigonometric functions of the angle in the right triangle for figure 3-18, view A.

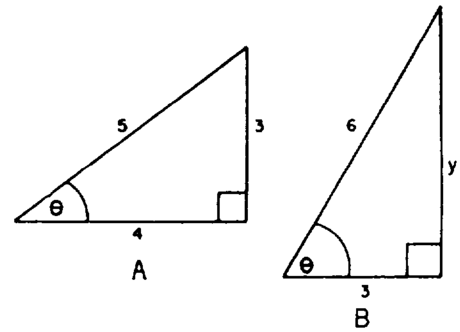


Figure 3-18.—Practice triangles.

SOLUTION:

$$\sin\theta = \frac{y}{r} = \frac{3}{5} = 0.6$$

$$\cos\theta = \frac{x}{r} = \frac{4}{5} = 0.8$$

$$\tan\theta = \frac{y}{x} = \frac{3}{4} = 0.75$$

$$\cot\theta = \frac{x}{y} = \frac{4}{3} = 1.33333 \text{ (rounded)}$$

$$\sec\theta = \frac{r}{x} = \frac{5}{4} = 1.25$$

$$\csc\theta = \frac{r}{y} = \frac{5}{3} = 1.66667 \text{ (rounded)}$$

EXAMPLE: Give the values of the trigonometric functions of the angle in the right triangle for figure 3-18, view B.

SOLUTION: Only two sides are given. To find the third side of the right triangle, use the Pythagorean theorem:

$$r^2 = x^2 + y^2$$

and

$$\begin{aligned} y^2 &= r^2 - x^2 \\ &= 6^2 - 3^2 \\ &= 36 - 9 \\ &= 27 \\ y &= \sqrt{27} \\ &= \sqrt{9 \cdot 3} \\ &= 3\sqrt{3} \end{aligned}$$

Now, using the values of x, y, and r, we find the values of the six trigonometric functions are as follows:

$$\sin \theta = \frac{y}{r} = \frac{3\sqrt{3}}{6} = \frac{\sqrt{3}}{2} = 0.86603 \text{ (rounded)}$$

$$\cos \theta = \frac{x}{r} = \frac{3}{6} = \frac{1}{2} = 0.5$$

$$\tan \theta = \frac{y}{x} = \frac{3\sqrt{3}}{3} = \sqrt{3} = 1.73205 \text{ (rounded)}$$

$$\cot \theta = \frac{x}{y} = \frac{3}{3\sqrt{3}} = \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{3} = 0.57735 \text{ (rounded)}$$

$$\sec \theta = \frac{r}{x} = \frac{6}{3} = 2$$

$$\csc \theta = \frac{r}{y} = \frac{6}{3\sqrt{3}} = \frac{2}{\sqrt{3}} = \frac{2\sqrt{3}}{3} = 1.15470 \text{ (rounded)}$$

TABLES OF TRIGONOMETRIC FUNCTIONS

Tables of trigonometric functions give the numerical values of the ratios of the sides of a right triangle that correspond to the trigonometric functions. Appendixes II and III are tables of trigonometric functions. These tables give values rounded to five decimal places of trigonometric functions for each minute from 0° to 90°. *Appendix II consists of tables of natural sines and cosines. Appendix III consists of tables of natural tangents and cotangents.*

For example, if we wanted to find sin 3° 25′, we would use appendix II, Natural Sines and Cosines, to first locate 3° on the first row of the table. Next, we would locate sin under 3° on the second row. Then, we would locate 25 along the first column of the table. Now, reading left to right across from 25 and from top to bottom under sin 3°, we find sin 3° 25′ = 0.05960. If we wanted to find cos 86° 35′, we would first locate 86° on the last row of the table. (The degrees on the top row range from 0° to 44°, and the degrees on the last row range from 45° to 90°.) Next, we would locate cos above 86° on the next to the last row. Then, we would locate 35 along the last column of the table. Now, reading right to left across from 35 and from bottom to top above cos 86°, we find cos 86° 35′ = 0.05960. Note that sin 3° 25′ = 0.05960 = cos 86° 35′. The reason for this will be discussed in chapter 4.

The tables in appendix III, Natural Tangents and Cotangents, are arranged in the same format as the tables in appendix II and are used in the same way. NOTE: Scientific calculators will give you the same values rounded to five decimal places as supplied in the tables in appendixes II and III.

Most tables list the sine, cosine, tangent, and cotangent of angles from 0° to 90°. Very few give the secant and cosecant since these functions of an angle are seldom used. When needed, they may be found from the values of the sine and cosine as follows:

$$\sec \theta = \frac{r}{x} = \frac{1}{\frac{x}{r}} = \frac{1}{\cos \theta}$$

and

$$\csc \theta = \frac{r}{y} = \frac{1}{\frac{y}{r}} = \frac{1}{\sin \theta}$$

Hence, the reciprocal of the secant function is the cosine function, and the reciprocal of the cosecant function is the sine function.

The tangent and cotangent functions may also be expressed in terms of the sine and cosine functions as follows:

$$\tan \theta = \frac{y}{x} = \frac{\frac{y}{r}}{\frac{x}{r}} = \frac{\sin \theta}{\cos \theta}$$

and

$$\cot \theta = \frac{x}{y} = \frac{\frac{x}{r}}{\frac{y}{r}} = \frac{\cos \theta}{\sin \theta}$$

In addition, the cotangent function may be determined as the reciprocal of the tangent function as follows:

$$\cot \theta = \frac{x}{y} = \frac{1}{\frac{y}{x}} = \frac{1}{\tan \theta}$$

NOTE: These relationships are the fundamental trigonometric identities that will be used extensively in solving more complex identities in chapter 6.

USE OF TRIGONOMETRIC RATIOS AND FUNCTIONS

The trigonometric ratios and trigonometric functions furnish powerful tools for use in problem solving of right triangles. Finding the remaining parts of a right triangle is possible if, in addition to the right angle, the length of one side and the length of any other side or the value of one of the acute angles is known.

EXAMPLE: Find the length of side y in figure 3-19, view A.

SOLUTION: We can use

$$\tan \theta = \frac{y}{x}$$

since we know one side and one angle. Thus,

$$\tan 35^\circ = \frac{y}{20}$$

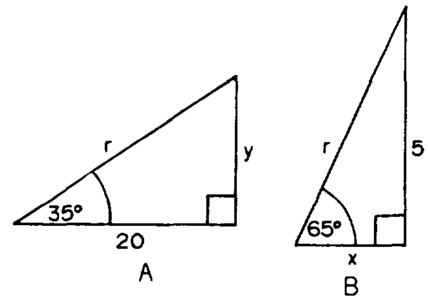


Figure 3-19.—Practical use of ratios.

From appendix III (or calculator), we find that

$$\tan 35^\circ = 0.70021$$

So,

$$0.70021 = \frac{y}{20}$$

$$y = (0.70021)(20)$$

$$= 14.0042$$

We could have also used cos θ, cot θ, or sec θ to find side y.

EXAMPLE: Find the value of r in figure 3-19, view B.

SOLUTION:

$$\sin \theta = \frac{y}{r}$$

$$\sin 65^\circ = \frac{5}{r}$$

$$r = \frac{5}{\sin 65^\circ}$$

$$r = \frac{5}{0.90631}$$

$$= 5.51688$$

We could have also used csc θ to find side y.

PRACTICE PROBLEMS:

Refer to figure 3-20 in working problems 1 through 4.

1. Find the values of the trigonometric functions of angle θ for the right triangle in view A.

2. Find the value of side y in view B using the sine function.

3. Find the value of side x in view C using the cosine function.

4. Find the value of side y in view D using the tangent function.

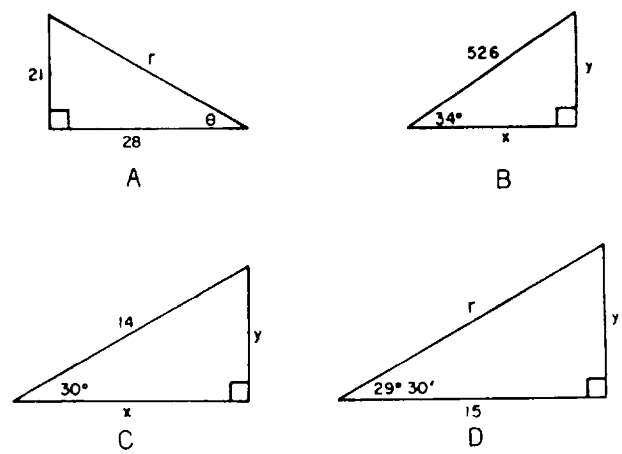


Figure 3-20.—Triangles for practice problems.

ANSWERS:

1. $\sin \theta = 21/35 = 3/5 = 0.6$

   $\cos \theta = 28/35 = 4/5 = 0.8$

   $\tan \theta = 21/28 = 3/4 = 0.75$

   $\cot \theta = 28/21 = 4/3 = 1.33333$

   $\sec \theta = 35/28 = 5/4 = 1.25$

   $\csc \theta = 35/21 = 5/3 = 1.66667$

2. 294.13394

3. 12.12442

4. 8.48655

SUMMARY

The following are the major topics covered in this chapter:

1. **Terminology:**

   Radius vector—The line that is rotated to generate an angle.

   Initial position—The original position of the radius vector.

   Terminal position—The final position of the radius vector.

   Positive angle—The angle generated by rotating the radius vector counterclockwise from the initial position.

   Negative angle—The angle generated by rotating the radius vector clockwise from the initial position.

2. **Degrees:** The degree system is the most common system of angular measurement. In this system a complete revolution is divided into 360 equal parts called *degrees.*

   $1 \text{ revolution} = 360°$

   $1° = 60'$

   $1' = 60''$

   For convenience, the 360° are divided into four equal parts of 90° each called *quadrants.*

   If $0° < \theta < 90°$, then θ is in quadrant I.

   If $90° < \theta < 180°$, then θ is in quadrant II.

   If $180° < \theta < 270°$, then θ is in quadrant III.

   If $270° < \theta < 360°$, then θ is in quadrant IV.

   If $\theta > 360°$, then θ lies in the same quadrant as $\theta - n(360°)$, where $n = 1, 2, 3, \ldots$ and $n(360°) < \theta$.

3. **Radians:** An even more fundamental method of angular measurement involves the *radian.* A *radian* is defined as an angle that, if its vertex is placed at the center of a circle, intercepts an arc equal in length to the radius of the circle.

   $2\pi \text{ radians} = 360°$

   $\pi \text{ radians} = 180°$

$$1 \text{ radian} = \frac{180^\circ}{\pi}$$

$$1^\circ = \frac{\pi}{180^\circ} \text{ radians}$$

The radian measure of an angle, θ, is the ratio of the length of the arc, s, it subtends to the length of the radius vector, r, of the circle in which it is the central angle or

$$\theta = \frac{s}{r}$$

4. **Other frequently used relationships between radians and degrees:**

Radians	Degrees
$\pi/6$	30
$\pi/4$	45
$\pi/3$	60
$\pi/2$	90
π	180
$3\pi/2$	270
2π	360

5. **Length of arc:**

$$s = \theta r$$

where θ represents the number of radians in a central angle, r the length of the radius of the circle, and s the length of the intercepted arc.

6. **Angular velocity:**

$$\omega = \frac{\theta}{t}$$

where θ is measured in radians and t is the unit time.

7. **Linear velocity:**

$$v = \frac{d}{t}$$

where d is the distance and t is the unit time.

$$v = r\omega$$

where r is the radius and ω is the angular velocity.

8. **Area of a sector of a circle:**

$$A = \frac{1}{2}r^2\theta$$

where θ is expressed in radians.

$$A = \frac{1}{2}rs$$

where r is the radius and s is the arc length.

9. **Mils:** The *mil* is a unit of small angular measurement that has military applications. The *mil* is defined as follows:

   1. 1/6,400 of the circumference of a circle.

      $$360° = 6{,}400 \text{ mils}$$

      $$1° = \frac{160}{9} \text{ mils}$$

      $$1 \text{ mil} = \frac{9°}{160}$$

   2. The angle subtended by an object 1 unit long, perpendicular to the line of sight, at a distance of 1,000 units.

      $$1 \text{ mil} \approx \frac{1}{1{,}000} \text{ radians}$$

      $$1 \text{ radian} \approx 1{,}000 \text{ mils}$$

10. **Pythagorean theorem:** The *Pythagorean theorem* states that in a right triangle, the square of the length of the hypotenuse, r, is equal to the sum of the squares of the lengths of the other two sides, x and y, or

$$r^2 = x^2 + y^2$$

11. **Similar triangles:** Whenever the angles of one triangle are equal to the corresponding angles in another triangle, the two triangles are said to be *similar* and the following proportions involving the lengths of their corresponding sides are true:

$$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$$

12. **Similar right triangles:** Two right triangles are similar if an acute angle of one triangle is equal to an acute angle of the other triangle. The following proportions involving the lengths of their corresponding sides are true:

$$\frac{x}{x'} = \frac{y}{y'} = \frac{r}{r'}$$

13. **Trigonometric ratios and functions:**

$$\sin\theta = \frac{y}{r} = \frac{\text{opposite}}{\text{hypotenuse}}$$

$$\cos\theta = \frac{x}{r} = \frac{\text{adjacent}}{\text{hypotenuse}}$$

$$\tan\theta = \frac{y}{x} = \frac{\text{opposite}}{\text{adjacent}}$$

$$\cot\theta = \frac{x}{y} = \frac{\text{adjacent}}{\text{opposite}}$$

$$\sec\theta = \frac{r}{x} = \frac{\text{hypotenuse}}{\text{adjacent}}$$

$$\csc\theta = \frac{r}{y} = \frac{\text{hypotenuse}}{\text{opposite}}$$

14. **Tables of trigonometric functions:** Tables of trigonometric functions give the numerical values of the ratios of the sides of a right triangle that correspond to the trigonometric functions. Appendix II consists of tables of natural sines and and cosines. Appendix III consists of tables of natural tangents and cotangents.

ADDITIONAL PRACTICE PROBLEMS

1. In which quadrant is the angle 5,370°?
2. Find the radian measure of the central angle in a circle with radius π inches if the angle subtends an arc of $3\pi/5$ inches.
3. Express 4,320° in radians, using π in the answer.
4. Express $11\pi/12$ in degrees.
5. If the length of the radius of a circle is 5 meters, find the length of arc subtended by a central angle with measure π radians.
6. Kim and Tom are riding on a Ferris wheel. Kim observes that it takes 30 seconds to make a complete revolution. Their seat is 35 feet from the axle of the wheel.

   a. What is their angular velocity in radians per second?

   b. What is their linear velocity in feet per minute?
7. Find the area of a sector of a circle if its central angle is 45° and the diameter of the circle is 28 centimeters.
8. Convert 17 7/9 mils to degrees.
9. Convert 3.6 degrees to mils.
10. Convert 9/5 mils to an approximate radian measure.
11. Convert 0.00145 radians to an approximate measurement in mils.
12. An airplane with a wing span of 84 feet is flying toward an observer. What is the distance of the plane from the observer when the plane subtends 7 mils?
13. The length of the hypotenuse of a right triangle is 17, and the length of one of the other sides is 8. What is the length of the remaining side?
14. Assume similar right triangles A and B have sides x, y, r, and x', y', r', respectively. If $x = 6$, $y = 8$, $r = 10$, and $y' = 1/2$, what are the values of x' and r'?
15. Find the values of the trigonometric functions θ of in a right triangle if the hypotenuse is 25 and the side adjacent to θ is 24.
16. If in a right triangle one of the acute angles is 56° 17′ and the hypotenuse is 10, what are the lengths of the other two sides?

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. 4th

2. 3/5

3. 24π

4. $165°$

5. 5π meters

6. a. $\pi/15$ radians per second

   b. 140π feet per minute

7. $49\pi/2$ square centimeters

8. $1°$

9. 64 mils

10. 0.0018 radians

11. 1.45 mils

12. 12,000 feet

13. 15

14. $x' = 3/8$

    $z' = 5/8$

15. $\sin\theta = 7/25 = 0.28$

    $\cos\theta = 24/25 = 0.96$

    $\tan\theta = 7/24 = 0.29167$ (rounded)

    $\cot\theta = 24/7 = 3.42857$ (rounded)

    $\sec\theta = 25/24 = 1.04167$ (rounded)

    $\csc\theta = 25/7 = 3.57143$ (rounded)

16. 5.5509 and 8.3179

CHAPTER 4

TRIGONOMETRIC ANALYSIS

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Use the rectangular coordinate system to determine the algebraic signs and the values of the trigonometric functions and to locate and define the trigonometric functions.
2. Relate any angle in standard position to its reference angle.
3. Determine the trigonometric functions of an angle in any quadrant, of negative angles, of coterminal angles, of frequently used angles, and of quadrantal angles.
4. Express the trigonometric functions of an angle in terms of their complement.
5. Recognize characteristics of the graphs of the sine, cosine, and tangent functions.

INTRODUCTION

This chapter is a continuation of the broad topic of trigonometry introduced in chapter 3. The topic is expanded in this chapter to allow analysis of angles greater than 90°. The chapter is extended as a foundation for analysis of the generalized angle; that is, an angle of any number of degrees. Additionally, the chapter introduces the concept of both positive and negative angles.

RECTANGULAR COORDINATE SYSTEM

The rectangular, or Cartesian, coordinate system introduced in *Mathematics*, Volume 1, was used in solving equations; in this chapter it is used to analyze the generalized angle. The following is a brief review of the rectangular coordinate system:

1. The vertical axis (*Y* axis in fig. 4-1) is considered positive above the origin and negative below the origin.

2. The horizontal axis (*X* axis in fig. 4-1) is positive to the right of the origin and negative to the left of the origin.

3. A point, $P(x,y)$, anywhere in a rectangular coordinate system may be located by two numbers. The value of x is called the *abscissa*. The value of y is called the *ordinate*. The abscissa and ordinate of a point are its *coordinates*.

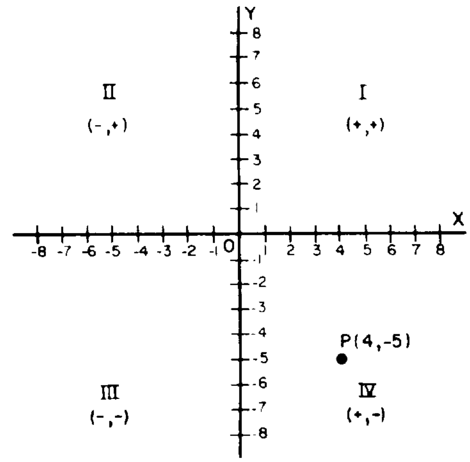


Figure 4-1.—Rectangular coordinate system.

4. In notation used to locate points, the coordinates are conventionally placed in parentheses and separated with a comma, with the abscissa always written first. The general form of this notation is $P(x,y)$. Thus, point P in figure 4-1 would have the notation $P(4, -5)$.

5. The quadrants are numbered in the manner described in chapter 3 of this course (shown as Roman numerals in figure 4-1).

6. The x coordinate is positive in the first (I) and fourth (IV) quadrants and negative in the second (II) and third (III) quadrants. The y coordinate is positive in the first and second quadrants and negative in the third and fourth quadrants. The signs of the coordinates are shown in parentheses in figure 4-1. The algebraic signs of the coordinates of a point are used in this chapter for determining the algebraic signs of trigonometric functions.

ANGLES IN STANDARD POSITION

To construct an *angle in standard position*, first lay out a rectangular coordinate system. Then draw the angle, θ, so that its vertex is at the origin of the coordinate system and its initial or original side is lying along the positive X axis as shown in figure 4-2. The terminal or final side of the angle will lie in any of the quadrants or on one of the axes separating the quadrants. When the terminal side falls on an axis, the angle is called a *quadrantal angle*, which will be discussed later in this chapter. In figure 4-2 the terminal side lies in quadrant II.

The quadrant in which an angle lies is determined by the terminal side. When an angle is placed in standard position, the angle is said to lie in the quadrant containing the terminal side. For example, the negative angle, θ, shown in standard position in figure 4-3, is said to lie in the second quadrant.

When two or more angles in standard position have their terminal sides located at the same position, they are said to be *coterminal*. If θ is any general angle, then θ plus or minus an integral multiple of 360° yields a coterminal angle.

For example, the angles θ, ϕ, and α in figure 4-4 are said to be coterminal angles. If

$$\theta = 45°$$

then

$$\phi = \theta - 360°$$

$$= 45° - 360°$$

$$= -315°$$

and

$$\alpha = \theta + 360°$$

$$= 45° + 360°$$

$$= 405°$$

The relationship of coterminal angles can be stated in a general form. *For any angle θ measured in degrees, any angle ϕ coterminal with θ can be found by*

$$\phi = \theta + n(360°)$$

where n is any integer (positive, negative, or zero); that is,

$$n = 0, \pm 1, \pm 2, \pm 3, \ldots$$

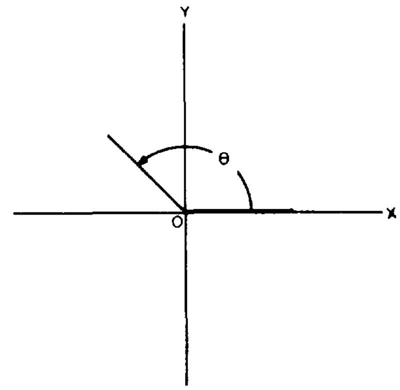


Figure 4-2.—Angle in standard position.

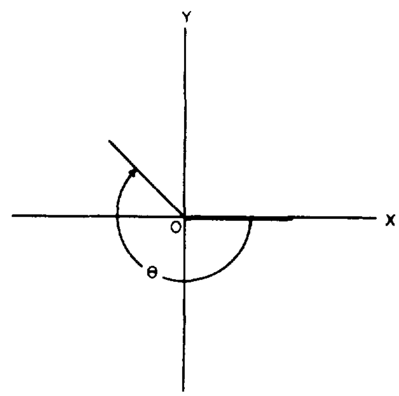


Figure 4-3.—Negative angle in quadrant II.

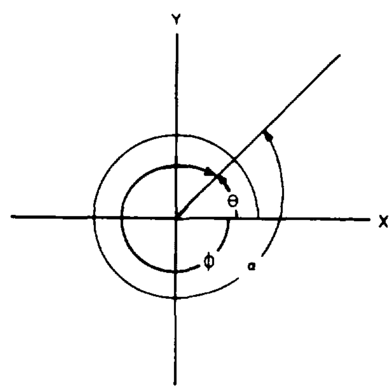


Figure 4-4.—Coterminal angles.

The principle of coterminal angles is used in developing other trigonometric relationships and other phases of trigonometric analysis. An expansion of this principle, discussed later in this chapter, states that the trigonometric functions of coterminal angles have the same value.

PRACTICE PROBLEMS:

Determine whether or not the following sets of angles are coterminal:

1. $60°$, $-300°$, $420°$

2. $0°$, $360°$, $180°$

3. $45°$, $-45°$, $345°$

4. $735°$, $-345°$, $-705°$

ANSWERS:

1. Coterminal

2. Not coterminal

3. Not coterminal

4. Coterminal

DEFINITIONS OF THE TRIGONOMETRIC FUNCTIONS

So far, the trigonometric functions have been defined as follows:

1. By labeling the sides of a right triangle x, y, and r.
2. By naming the sides of a right triangle adjacent, opposite, and hypotenuse.

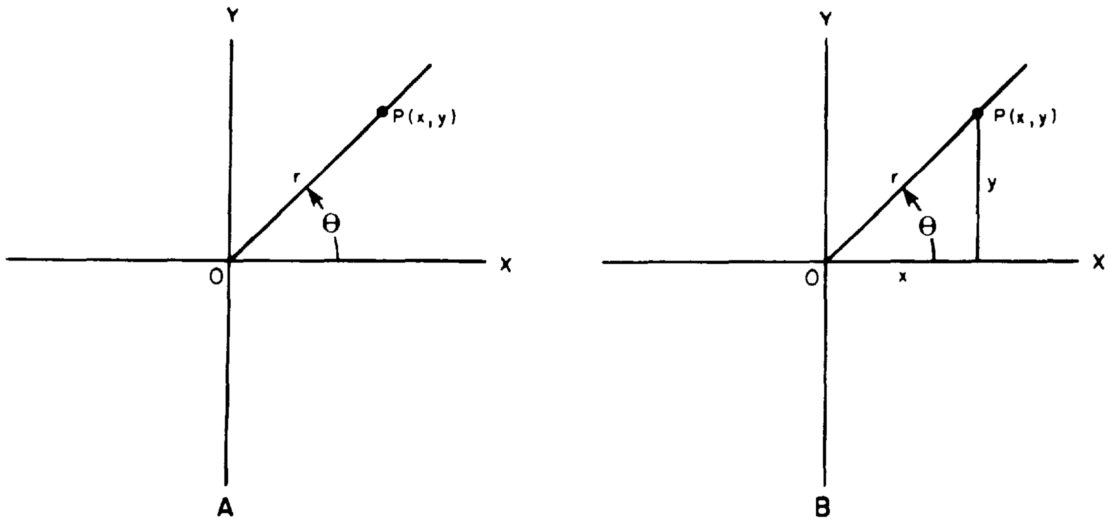


Figure 4-5.—Functions of general angles.

In this chapter we will introduce a third set of definitions using the nomenclature of the coordinate system. Note that each definition defines the same functions using different terminology.

To arrive at the third set of definitions, construct an angle in standard position on a coordinate system as shown in figure 4-5, view A. Choose point $P(x,y)$ on the final position of the radius vector. Distance OP is denoted by the positive number r for the length of the radius.

By constructing a right triangle using $P(x,y)$ and r, as in figure 4-5, view B, the six trigonometric functions are classified as follows:

$$\sin \theta = \frac{y}{r} = \frac{\text{ordinate}}{\text{length of radius}}$$

$$\cos \theta = \frac{x}{r} = \frac{\text{abscissa}}{\text{length of radius}}$$

$$\tan \theta = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}$$

$$\cot \theta = \frac{x}{y} = \frac{\text{abscissa}}{\text{ordinate}}$$

$$\sec \theta = \frac{r}{x} = \frac{\text{length of radius}}{\text{abscissa}}$$

$$\csc \theta = \frac{r}{y} = \frac{\text{length of radius}}{\text{ordinate}}$$

The value of each function is dependent on angle θ and not on the selection of point $P(x,y)$. If a different point were chosen, the length of r, as well as the values of the x and y coordinates, would change proportionally, but the ratios would be unchanged.

EXAMPLE: Find the sine and cosine of angle θ in figure 4-5, view A, for the point $P(3,4)$.

SOLUTION: To determine the sine and cosine of θ, we must find the value of r. Since the values of the x and y coordinates correspond to the lengths of the sides x and y in figure 4-5, view B, we can determine the length of r by using the Pythagorean theorem or by recalling from *Mathematics*, Volume 1, the 3-4-5 triangle. In either case, the length of r is 5 units. Hence,

$$\sin\theta = \frac{\text{ordinate}}{\text{length of radius}}$$

$$= \frac{4}{5}$$

and

$$\cos\theta = \frac{\text{abscissa}}{\text{length of radius}}$$

$$= \frac{3}{5}$$

NOTE: For the remainder of this chapter, all angles are understood to be in standard position, unless otherwise stated.

PRACTICE PROBLEMS:

Find the sine, cosine, and tangent of the angles whose radius vectors pass through the following points:

1. $P(5,12)$
2. $P(1,1)$
3. $P(1, \sqrt{3})$
4. $P(3,2)$

ANSWERS:

1. $\sin \theta = 12/13$

   $\cos \theta = 5/13$

   $\tan \theta = 12/5$

2. $\sin \theta = 1/\sqrt{2} = \sqrt{2}/2$

   $\cos \theta = 1/\sqrt{2} = \sqrt{2}/2$

   $\tan \theta = 1/1 = 1$

3. $\sin \theta = \sqrt{3}/2$

   $\cos \theta = 1/2$

   $\tan \theta = \sqrt{3}/1 = \sqrt{3}$

4. $\sin \theta = 2/\sqrt{13} = 2\sqrt{13}/13$

   $\cos \theta = 3/\sqrt{13} = 3\sqrt{13}/13$

   $\tan \theta = 2/3$

QUADRANT SYSTEM

The quadrants formed in the rectangular coordinate system are used to determine the algebraic signs of the trigonometric functions. The quadrants in figure 4-6 show the algebraic signs of the trigonometric functions in the various quadrants.

In the first quadrant the abscissa and ordinate are always positive. The radius vector is always taken as positive. Therefore, *all the trigonometric ratios are positive for angles in the first quadrant. For angles in the second quadrant, only the ratios involving the ordinate and the radius vector are positive. These are the sine and cosecant ratios. For angles in the third quadrant, where the ordinate and abscissa are both negative, only the ratios involving the abscissa and the ordinate are positive.*

II	I
$\sin \theta = +/+ = +$	$\sin \theta = +/+ = +$
$\cos \theta = -/+ = -$	$\cos \theta = +/+ = +$
$\tan \theta = +/- = -$	$\tan \theta = +/+ = +$
$\cot \theta = -/+ = -$	$\cot \theta = +/+ = +$
$\sec \theta = +/- = -$	$\sec \theta = +/+ = +$
$\csc \theta = +/+ = +$	$\csc \theta = +/+ = +$

III	IV
$\sin \theta = -/+ = -$	$\sin \theta = -/+ = -$
$\cos \theta = -/+ = -$	$\cos \theta = +/+ = +$
$\tan \theta = -/- = +$	$\tan \theta = -/+ = -$
$\cot \theta = -/- = +$	$\cot \theta = +/- = -$
$\sec \theta = +/- = -$	$\sec \theta = +/+ = +$
$\csc \theta = +/- = -$	$\csc \theta = +/- = -$

Figure 4-6.—Signs of functions.

These are the tangent and cotangent ratios. For angles in the fourth quadrant, ratios involving the radius vector and the abscissa are positive. These are the cosine and the secant ratios.

NOTE: In each quadrant the sine and cosecant have the same sign, the cosine and the secant have the same sign, and the tangent and cotangent have the same sign.

The last group of practice problems involved angles in the first quadrant only, where all of the functions were positive. When an angle lies in one of the other quadrants, the trigonometric functions may be positive or negative.

EXAMPLE: Find all of the trigonometric functions of θ if $\tan \theta = 5/12$, $\sin \theta < 0$, and $r = 13$.

SOLUTION: Reference to figure 4-6 shows that an angle with a positive tangent and a negative sine can only occur in the third quadrant. The point in the third quadrant has coordinates $(-12, -5)$. (See fig. 4-7)

We can now read the trigonometric ratios from the figure:

$$\sin \theta = \frac{\text{ordinate}}{\text{length of radius}} = \frac{-5}{13}$$

$$\cos \theta = \frac{\text{abscissa}}{\text{length of radius}} = \frac{-12}{13}$$

$$\tan \theta = \frac{\text{ordinate}}{\text{abscissa}} = \frac{-5}{-12} = \frac{5}{12}$$

$$\cot \theta = \frac{\text{abscissa}}{\text{ordinate}} = \frac{-12}{-5} = \frac{12}{5}$$

$$\sec \theta = \frac{\text{length of radius}}{\text{abscissa}} = \frac{13}{-12} = \frac{-13}{12}$$

$$\csc \theta = \frac{\text{length of radius}}{\text{ordinate}} = \frac{13}{-5} = \frac{-13}{5}$$

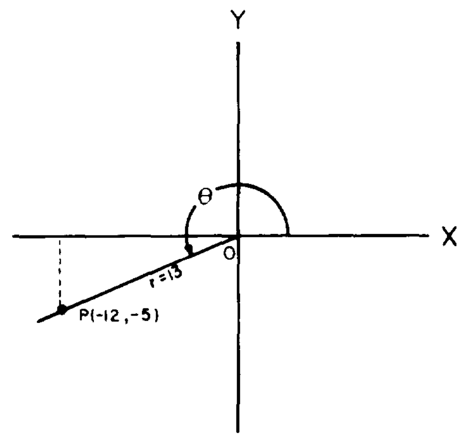


Figure 4-7.—Finding the trigonometric functions for a third-quadrant angle.

EXAMPLE: Find all of the trigonometric functions of θ if $\csc \theta = -17/15$ and $\cos \theta < 0$.

SOLUTION: The cosecant is negative in the same quadrants as the sine; that is, quadrants III and IV. The cosine is negative in quadrants II and III. Therefore, the cosecant and cosine are both negative in quadrant III. (Refer to fig. 4-6.) The ordinate in the third quadrant is -15 and the radius is 17.

NOTE: The fraction $-17/15$ indicates that either the numerator or denominator is negative, but not both. In this case, we know that the ordinate (denominator) is negative since the radius (numerator) is always positive.

From the Pythagorean theorem the abscissa in the third quadrant is

$$x^2 = r^2 - y^2$$

$$= (17)^2 - (-15)^2$$

$$= 289 - 225$$

$$= 64$$

$$x = -8$$

Therefore, referring to figure 4-8, the six trigonometric functions are as follows:

$$\sin\theta = -15/17$$

$$\cos\theta = -8/17$$

$$\tan\theta = -15/-8 = 15/8$$

$$\cot\theta = -8/-15 = 8/15$$

$$\sec\theta = 17/-8 = -17/8$$

$$\csc\theta = 17/-15 = -17/15$$

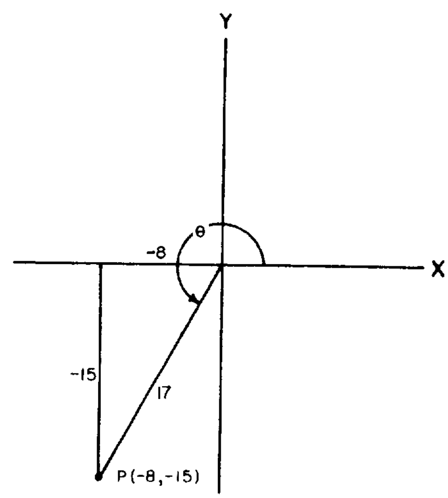


Figure 4-8.—Construction of a triangle in quadrant 3.

EXAMPLE: If $\sec\theta = -25/24$ and $\tan\theta = -7/24$, find the other four trigonometric ratios of θ.

SOLUTION: The secant and tangent are both negative in the second quadrant. In the second quadrant the abscissa is -24, the ordinate is 7, and the radius is 25 (refer to fig. 4-9); so,

$$\sin \theta = 7/25$$

$$\cos \theta = -24/25$$

$$\cot \theta = -24/7$$

$$\csc \theta = 25/7$$

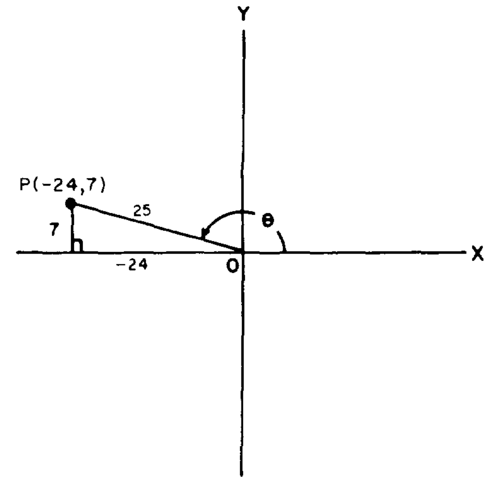


Figure 4-9.—Construction of a triangle in quadrant 2.

PRACTICE PROBLEMS:

Without using tables, find the six trigonometric functions of θ under the following conditions:

1. $\tan \theta = 3/4$, $r = 5$, and θ is not in the first quadrant.
2. $\tan \theta = -21/20$, $r = 29$, and $\cos \theta > 0$.
3. $\cos = -3/5$ and $\cot \theta = 3/4$.
4. $\tan \theta = -8/15$ and $\csc \theta$ is positive.

Indicate the quadrant in which the terminal side of θ lies for the following conditions:

5. $\sin \theta > 0$ and $\cos \theta < 0$
6. $\cos \theta < 0$ and $\csc \theta < 0$
7. $\sec \theta > 0$ and $\cot \theta < 0$

ANSWERS:

1. $\sin \theta = -3/5$

   $\cos \theta = -4/5$

   $\tan \theta = -3/-4 = 3/4$

   $\cot \theta = -4/-3 = 4/3$

   $\sec \theta = 5/-4 = -5/4$

   $\csc \theta = 5/-3 = -5/3$

2. $\sin \theta = -21/29$

$\cos \theta = 20/29$

$\tan \theta = -21/20$

$\cot \theta = 20/-21 = -20/21$

$\sec \theta = 29/20$

$\csc \theta = 29/-21 = -29/21$

3. $\sin \theta = -4/5$

$\cos \theta = -3/5$

$\tan \theta = -4/-3 = 4/3$

$\cot \theta = -3/-4 = 3/4$

$\sec \theta = 5/-3 = -5/3$

$\csc \theta = 5/-4 = -5/4$

4. $\sin \theta = 8/17$

$\cos \theta = -15/17$

$\tan \theta = 8/-15 = -8/15$

$\cot \theta = -15/8$

$\sec \theta = 17/-15 = -17/15$

$\csc \theta = 17/8$

5. 2

6. 3

7. 4

REFERENCE ANGLE

The *reference angle*, θ', for any angle, θ, in standard position is the smallest positive angle between the radius vector of θ and the X axis, such that $0° \leq \theta' \leq 90°$. In general, the reference angle for θ is

$$\theta' = n(180°) \pm \theta$$

where n is any integer. Expressed in an equivalent form

$$\theta' = n\pi \pm \theta$$

where again n is any integer and $0 \leq \theta' \leq \pi/2$.

Refer to figure 4-10. If P is any point on the radius vector, a perpendicular from P to the point A on the X axis forms a right triangle with sides OA, AP, and OP. We call this triangle the *reference triangle*. The relationship between θ, θ', and the reference triangle in each quadrant is shown in figure 4-10.

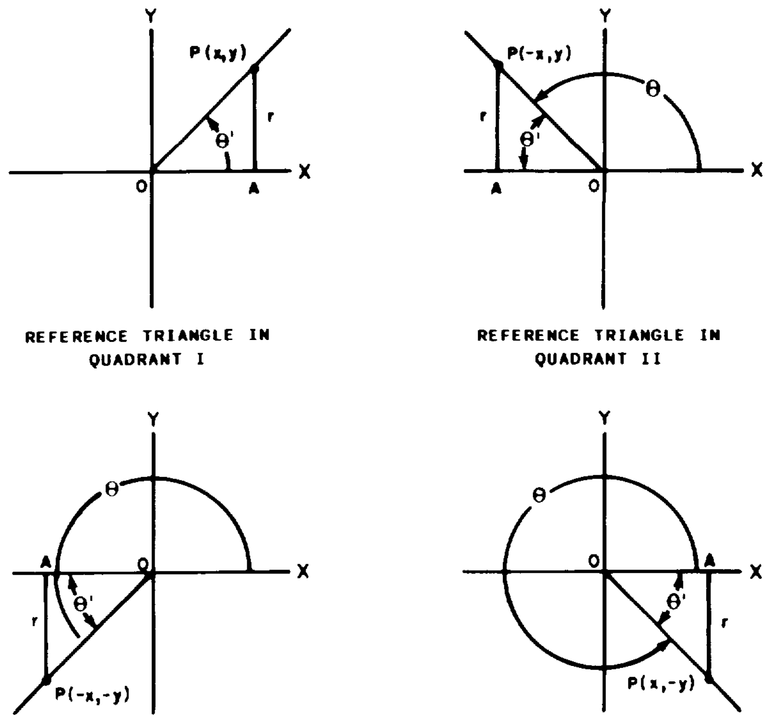


Figure 4-10.—Reference triangles in each quadrant.

FUNCTIONS OF ANGLES IN ANY QUADRANT

In addition to the reference triangle, formulas are used for determining the signs of the functions at any angle. These are called *reduction formulas*. This section shows the geometrical development of some of the most commonly used reduction formulas. In general, reduction formulas provide a means of reducing the functions of any angle to an equivalent expression for the function in terms of a positive acute angle, θ. The reduction formulas can be used in the solution of some trigonometric identities and in other applications requiring analysis of trigonometric functions.

The function of θ and the reduction formulas of the functions of $180° - \theta$, $180° + \theta$, and $360° - \theta$ are summarized in the following paragraphs according to their respective quadrants.

QUADRANT I

Any angle in the first quadrant can be represented by θ; that is,

$$\sin \theta = \frac{y}{r}$$

$$\cos \theta = \frac{x}{r}$$

$$\tan \theta = \frac{y}{x}$$

$$\cot \theta = \frac{x}{y}$$

$$\sec \theta = \frac{r}{x}$$

$$\csc \theta = \frac{r}{y}$$

QUADRANT II

Any angle in the second quadrant can be represented by $180° - \theta$; that is,

$$\sin (180° - \theta) = \frac{y}{r} = \sin \theta$$

$$\cos (180° - \theta) = -\frac{x}{r} = -\cos \theta$$

$$\tan (180° - \theta) = -\frac{y}{x} = -\tan \theta$$

$$\cot (180° - \theta) = -\frac{x}{y} = -\cot \theta$$

$$\sec (180° - \theta) = -\frac{r}{x} = -\sec \theta$$

$$\csc (180° - \theta) = \frac{r}{y} = \csc \theta$$

EXAMPLE: Use a reduction formula and appendix III to find the cotangent of 112°.

SOLUTION: Since 112° is in the second quadrant, where

$$\cot (180° - \theta) = -\cot \theta$$

then

$$\cot 112° = \cot (180° - 68°)$$

$$= -\cot 68°$$

$$= -0.40403$$

QUADRANT III

Any angle in the third quadrant can be represented by 180° + θ; that is,

$$\sin (180° + \theta) = -\frac{y}{r} = -\sin \theta$$

$$\cos (180° + \theta) = -\frac{x}{r} = -\cos \theta$$

$$\tan (180° + \theta) = \frac{y}{x} = \tan \theta$$

$$\cot (180° + \theta) = \frac{x}{y} = \cot \theta$$

$$\sec (180° + \theta) = -\frac{r}{x} = -\sec \theta$$

$$\csc (180° + \theta) = -\frac{r}{y} = -\csc \theta$$

EXAMPLE: Use a reduction formula and appendix II to find the sine of 220°.

SOLUTION: Since 220° is in the third quadrant, where

$$\sin (180° + \theta) = -\sin \theta$$

then

$$\sin 220° = \sin (180° + 40°)$$

$$= -\sin 40°$$

$$= -0.64279$$

QUADRANT IV

Any angle in the fourth quadrant can be represented by $360° - \theta$; that is,

$$\sin (360° - \theta) = -\frac{y}{r} = -\sin \theta$$

$$\cos (360° - \theta) = \frac{x}{r} = \cos \theta$$

$$\tan (360° - \theta) = -\frac{y}{x} = -\tan \theta$$

$$\cot (360° - \theta) = -\frac{x}{y} = -\cot \theta$$

$$\sec (360° - \theta) = \frac{r}{x} = \sec \theta$$

$$\csc (360° - \theta) = -\frac{r}{y} = -\csc \theta$$

EXAMPLE: Find cos 324°.

SOLUTION: Since

$$\cos (360° - \theta) = \cos \theta$$

then

$$\cos 324° = \cos (360° - 36°)$$

$$= \cos 36°$$

$$= 0.80902$$

FUNCTIONS OF NEGATIVE ANGLES

The following relationships enable us to change a function with a negative angle into the same function with a positive angle:

$$\sin(-\theta) = -\frac{y}{r} = -\sin\theta$$

$$\cos(-\theta) = \frac{x}{r} = \cos\theta$$

$$\tan(-\theta) = -\frac{y}{x} = -\tan\theta$$

$$\cot(-\theta) = -\frac{x}{y} = -\cot\theta$$

$$\sec(-\theta) = \frac{r}{x} = \sec\theta$$

$$\csc(-\theta) = -\frac{r}{y} = -\csc\theta$$

EXAMPLE: Find $\tan(-350°)$.

SOLUTION: Since

$$\tan(-\theta) = -\tan\theta$$

then

$$\tan(-350°) = -\tan 350°$$

and

$$\begin{aligned} -\tan 350° &= -\tan(360° - 10°) \\ &= -(-\tan 10°) \\ &= 0.17633 \end{aligned}$$

FUNCTIONS OF COTERMINAL ANGLES

For a *coterminal angle* in the form of

$$\theta' = n(360°) + \theta$$

where n is any integer θ and is an integral multiple of θ', the trigonometric functions of θ' are equal to those of θ. In other words, θ is the remainder obtained by dividing θ' by 360, and n is the number of times 360 will divide into θ'. Thus, we can find the ratios of a coterminal angle greater than 360° by dividing θ' by 360 and finding the functions of the remainder.

EXAMPLE: Find the cosine of −2,080°. (Refer to fig. 4-11.)

SOLUTION: Divide 2,080 by 360.

$$\begin{array}{r|l} & \ \ \ \ 5 \\ \hline 360 & 2{,}080 \\ & \underline{1{,}800} \\ & \ \ 280 \end{array}$$

So,

$$\cos(-2{,}080^\circ) = \cos(-280^\circ)$$

and

$$\begin{aligned}\cos(-280^\circ) &= \cos(280^\circ) \\ &= \cos(360^\circ - 80^\circ) \\ &= \cos 80^\circ \\ &= 0.17365\end{aligned}$$

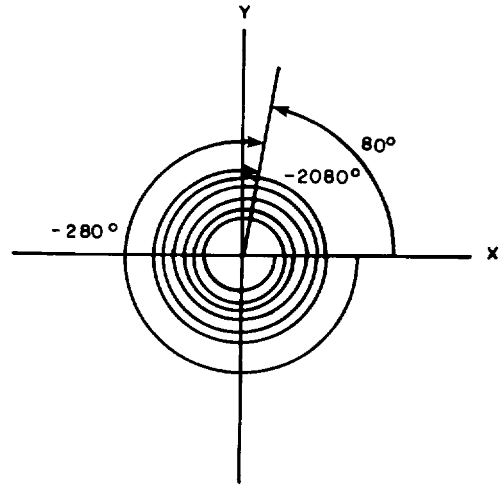


Figure 4-11.—Coterminal angles of −2,080°.

PRACTICE PROBLEMS:

Use reduction formulas and appendixes II and III to find the values of the sine, cosine, and tangent of θ given the following angles:

1. 137°
2. 214°
3. 325°
4. −70°
5. 1,554°

ANSWERS:

1. $\sin 137° = \sin 43° = 0.68200$

   $\cos 137° = -\cos 43° = -0.73135$

   $\tan 137° = -\tan 43° = -0.93252$

2. $\sin 214° = -\sin 34° = -0.55919$

   $\cos 214° = -\cos 34° = -0.82904$

   $\tan 214° = \tan 34° = 0.67451$

3. $\sin 325° = -\sin 35° = -0.57358$

   $\cos 325° = \cos 35° = 0.81915$

   $\tan 325° = -\tan 35° = -0.70021$

4. $\sin (-70°) = -\sin 70° = -0.93969$

   $\cos (-70°) = \cos 70° = 0.34202$

   $\tan (-70°) = -\tan 70° = -2.74748$

5. $\sin 1{,}554° = \sin 114° = \sin 66° = 0.91355$

   $\cos 1{,}554° = \cos 114° = -\cos 66° = -0.40674$

   $\tan 1{,}554° = \tan 114° = -\tan 66° = -2.24604$

COFUNCTIONS AND COMPLEMENTARY ANGLES

Complementary angles are angles whose sum is 90°. Two trigonometric functions that have equal values for complementary angles are called *cofunctions*.

Inspect the triangle in figure 4-12. We will compare the six trigonometric functions of θ with the six trigonometric functions of $90° - \theta$.

Functions of θ	Functions of $90° - \theta$
$\sin \theta = \frac{y}{r}$	$\cos (90° - \theta) = \frac{y}{r}$
$\cos \theta = \frac{x}{r}$	$\sin (90° - \theta) = \frac{x}{r}$
$\tan \theta = \frac{y}{x}$	$\cot (90° - \theta) = \frac{y}{x}$
$\cot \theta = \frac{x}{y}$	$\tan (90° - \theta) = \frac{x}{y}$
$\sec \theta = \frac{r}{x}$	$\csc (90° - \theta) = \frac{r}{x}$
$\csc \theta = \frac{r}{y}$	$\sec (90° - \theta) = \frac{r}{y}$

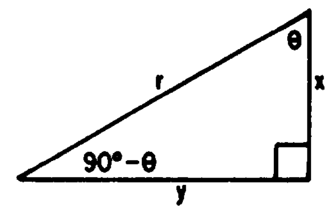


Figure 4-12.—Complementary angles.

We see from the above relationships that

$$\sin \theta = \cos (90° - \theta)$$

$$\cos \theta = \sin (90° - \theta)$$

$$\tan \theta = \cot (90° - \theta)$$

$$\cot \theta = \tan (90° - \theta)$$

$$\sec \theta = \csc (90° - \theta)$$

$$\csc \theta = \sec (90° - \theta)$$

Hence, a trigonometric function of an angle is equal to the confunction of its complement.

NOTE: These relationships may explain to you how the cosine, cotangent, and cosecant functions received their names.

The confunction principle accounts for the format of the tables of trigonometric functions in appendixes II and III. For example, in appendix II

$$\sin 21° 30' = 0.36650$$

and

$$\cos 68^\circ 30' = 0.36650$$

Notice that

$$21^\circ 30' + 68^\circ 30' = 90^\circ$$

PRACTICE PROBLEMS:

Express the following as a function of the complementary angle:

1. $\sin 27^\circ$
2. $\tan 38^\circ 17'$
3. $\csc 41^\circ$
4. $\cos 16^\circ 30' 22''$
5. $\sec 79^\circ 37' 16''$
6. $\cos 56^\circ$
7. $\cot 48^\circ$

ANSWERS:

1. $\cos 63^\circ$
2. $\cot 51^\circ 43'$
3. $\sec 49^\circ$
4. $\sin 73^\circ 29' 38''$
5. $\csc 10^\circ 22' 44''$
6. $\sin 34^\circ$
7. $\tan 42^\circ$

SPECIAL ANGLES

Two groups of angles are discussed in this section. The first group of angles is considered because the angles can be determined geometrically and are used frequently in problem solving. The second group is considered because the radius vectors of the angles fall on one of the coordinate axes, not in one of the quadrants.

FREQUENTLY USED ANGLES

As stated previously, the approximate values of the trigonometric functions for any angle can be read directly from tables or can be determined from tables by the use of the principles stated in this text. However, certain frequently used simple angles exist for which the exact function values are often used because these exact values can easily be determined geometrically. In the following paragraphs the geometrical determination of these functions is shown.

30°-60° Angles

The trigonometric functions of 30° and 60° can be determined geometrically. Construct an equilateral triangle with side lengths of 2 units, such as triangle *OYA* in figure 4-13. (The functions to be determined are not dependent on the lengths of the sides being 2 units; this size was selected for convenience.)

Drop a perpendicular from angle *Y* to the base of the triangle at point *X*. The right triangles *YXO* and *YXA* are formed by the perpendicular, which also bisects angle *Y* forming a 30° angle. Moreover, since side *OA* is 2 units long, then *OX* is 1 unit long and *YX* is $\sqrt{3}$ units long (using the Pythagorean theorem).

Figures 4-14 and 4-15 show a 30° and a 60° reference triangle, respectively. From these figures we can determine the trigonometric ratios of 30° and 60°, which are summarized in table 4-1.

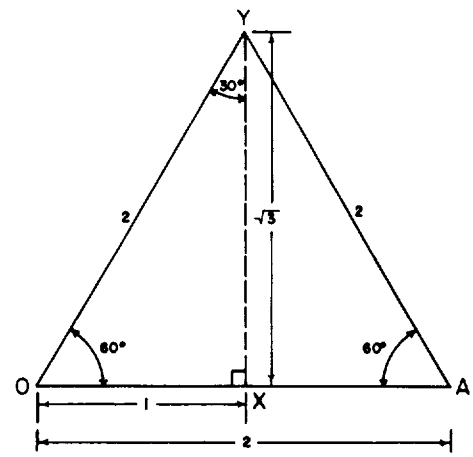


Figure 4-13.—Geometrical construction of 30° and 60° right triangles.

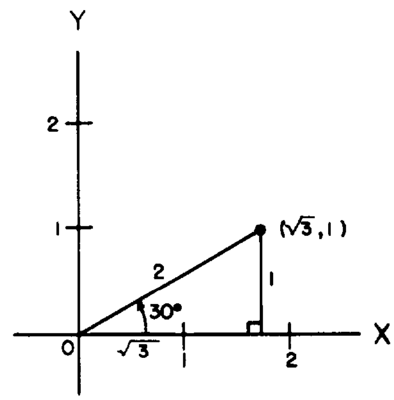


Figure 4-14.—30° reference triangle.

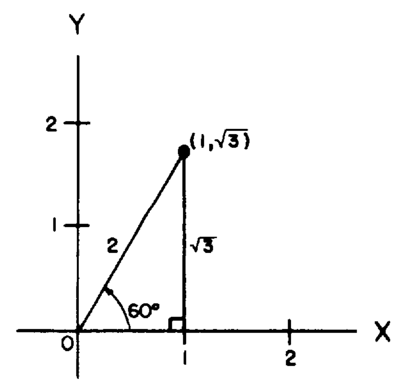


Figure 4-15.—60° reference triangle.

Table 4-1.—Trigonometric Functions of Special Angles

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
30°	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
60°	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
45°	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$

NOTE: The values of the confunctions interchange since 30° and 60° are complementary angles. An easy way to recall the values of the functions of right triangles with 30° and 60° complementary angles is to remember that the ratio of the sides is always 1, 2, and $\sqrt{3}$, where the largest side value represents the length of the hypotenuse.

EXAMPLE: Find the six trigonometric functions of 300°.

SOLUTION: Referring to figure 4-16, 300° is in the fourth quadrant and its reference angle is 60°. Therefore,

$$\sin 300^\circ = -\sqrt{3}/2$$

$$\cos 300^\circ = 1/2$$

$$\tan 300^\circ = -\sqrt{3}/1 = -\sqrt{3}$$

$$\cot 300^\circ = 1/-\sqrt{3} = -\sqrt{3}/3$$

$$\sec 300^\circ = 2/1 = 2$$

$$\csc 300^\circ = 2/-\sqrt{3} = -2\sqrt{3}/3$$

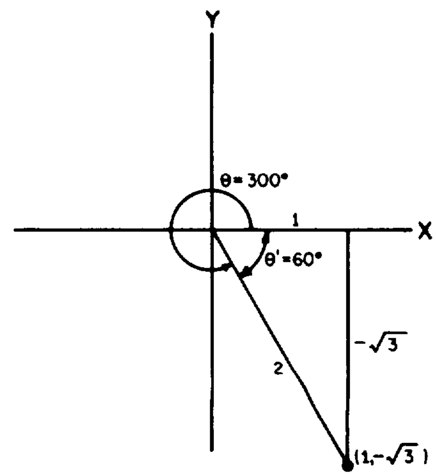


Figure 4-16.—300° angle in standard position.

45° Angles

Refer to figure 4-17. If one of the acute angles of the right triangle *OXY* is 45°, then the other acute angle is also 45°. Since triangle *OXY* is an isosceles triangle, then sides *OX* and *XY* are equal. If we let *OX* and *XY* be 1 unit long, then by the Pythagorean theorem, the length of *OY* is $\sqrt{2}$ units.

NOTE: This relationship is true of all 45° triangles and is not altered by the lengths of the legs. The ratio of the sides of right triangles with 45° complementary angles will always be 1, 1, and $\sqrt{2}$, where the largest value represents the length of the hypotenuse.

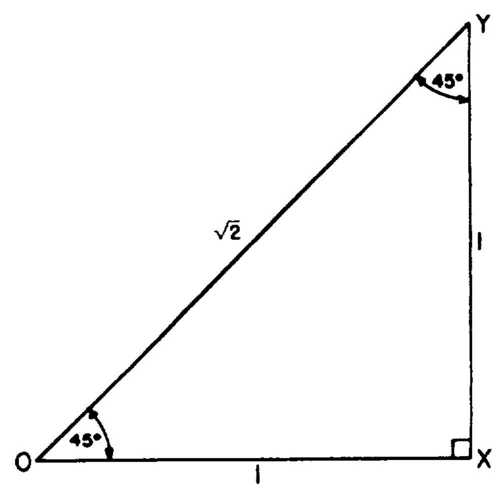


Figure 4-17.—Geometrical construction of a 45° right triangle.

Figure 4-18 shows a 45° reference triangle. From this figure we can determine the trigonometric ratios of 45°, which are also summarized in table 4-1.

EXAMPLE: Find the six trigonometric functions of 135°.

SOLUTION: Referring to figure 4-19, 135° is in the second quadrant and its reference angle is 45°. Therefore,

$$\sin 135° = 1/\sqrt{2} = \sqrt{2}/2$$

$$\cos 135° = -1/\sqrt{2} = -\sqrt{2}/2$$

$$\tan 135° = 1/-1 = -1$$

$$\cot 135° = -1/1 = -1$$

$$\sec 135° = \sqrt{2}/-1 = -\sqrt{2}$$

$$\csc 135° = \sqrt{2}/1 = \sqrt{2}$$

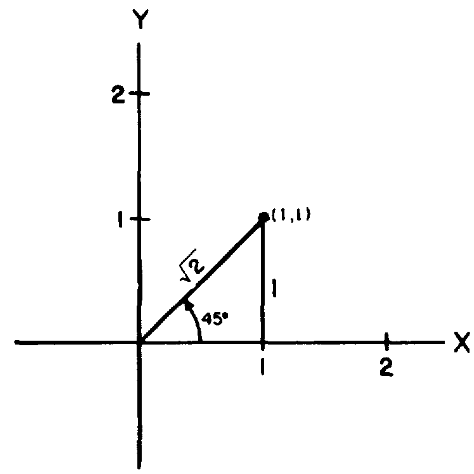


Figure 4-18.—45° reference triangle.

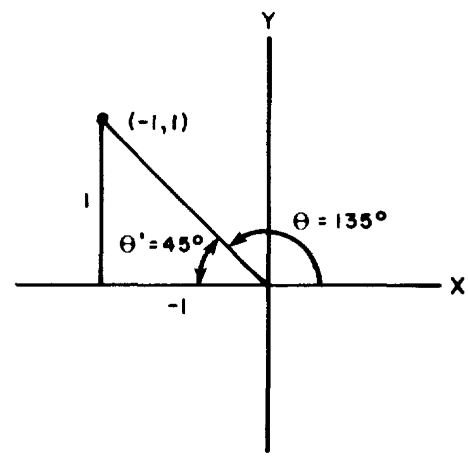


Figure 4-19.—135° angle in standard position.

QUADRANTAL ANGLES

An angle whose terminal side lies on a coordinate axis when the angle is in standard position is a *quadrantal angle*. Angles of 0°, ±90°, ±180°, and ±270° are quadrantal angles.

The trigonometric functions of the quadrantal angles are defined in the same manner as before, except for the restriction that a function is undefined when the denominator of the ratio is zero.

To derive the functions of the quandrantal angles, we choose points on the terminal sides, where $r = 1$, as shown in figure 4-20. Then either x or y is zero, and the other coordinate is either positive or negative 1.

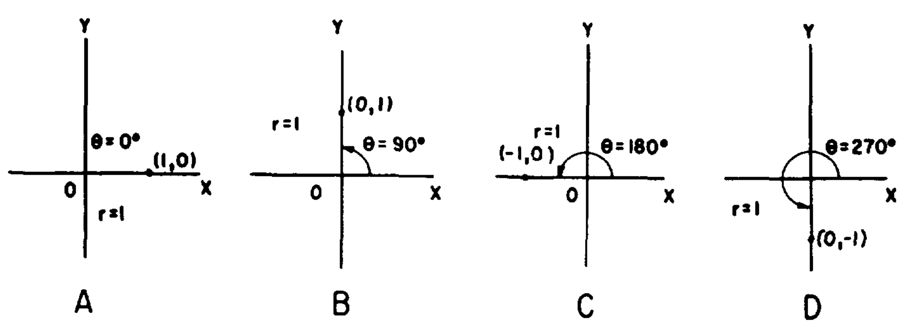


Figure 4-20.—Functions of quadrantal angles.

Table 4-2.—Functions of Quadrantal Angles

θ Deg.	θ Rad.	sin θ	cos θ	tan θ	cot θ	sec θ	csc θ
0°	0	0	1	0	undefined	1	undefined
90°	$\frac{\pi}{2}$	1	0	undefined	0	undefined	1
180°	π	0	-1	0	undefined	-1	undefined
270°	$\frac{3\pi}{2}$	-1	0	undefined	0	undefined	-1

Consider view C of figure 4-20 in which $\theta = 180°$. For the point $P(-1,0)$ and $r = 1$, we have

$$\sin 180° = 0/1 = 0$$

$$\cos 180° = -1/1 = -1$$

$$\tan 180° = 0/-1 = 0$$

$$\cot 180° = -1/0 \text{ (undefined)}$$

$$\sec 180° = 1/-1 = -1$$

$$\csc 180° = 1/0 \text{ (undefined)}$$

The values of the functions of the other quadrantal angles can be found by a similar procedure and are summarized in table 4-2.

EXAMPLE: Determine the six trigonometric functions of 990°.

SOLUTION: Referring to figure 4-21, we see that 990° lies on the same quadrantal axes as 270°. Therefore, for $P(0,-1)$ and $r = 1$, we have

$$\sin 990° = -1/1 = -1$$

$$\cos 990° = 0/1 = 0$$

$$\tan 990° = -1/0 \text{ (undefined)}$$

$$\cot 990° = 0/-1 = 0$$

$$\sec 990° = 1/0 \text{ (undefined)}$$

$$\csc 990° = 1/-1 = -1$$

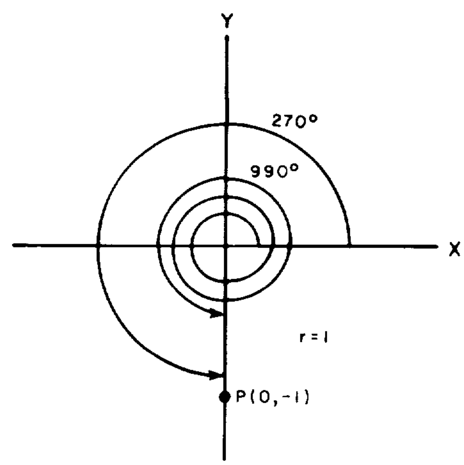


Figure 4-21.—990° angle.

PRACTICE PROBLEMS:

Without using the appendixes, determine the trigonometric functions of problems 1 through 5.

1. $\theta = 210°$

2. $\theta = 360°$

3. $\theta = 585°$

4. $\theta = -180°$

5. $\theta = -315°$

Without using the appendixes, evaluate problems 6 through 8. [Note that $\sin^2 \theta = (\sin \theta)^2$.]

6. $\sin^2 150° + \cos^2 150°$

7. $2 \sin 120° \cos 120°$

8. $\cos^2 135° - \sin^2 135°$

ANSWERS:

1. $\sin 210° = -1/2$

   $\cos 210° = -\sqrt{3}/2$

   $\tan 210° = -1/-\sqrt{3} = \sqrt{3}/3$

   $\cot 210° = -\sqrt{3}/-1 = \sqrt{3}$

   $\sec 210° = 2/-\sqrt{3} = -2\sqrt{3}/3$

   $\csc 210° = 2/-1 = -2$

2. $\sin 360° = 0/1 = 0$

   $\cos 360° = 1/1 = 1$

   $\tan 360° = 0/1 = 0$

   $\cot 360° = 1/0$ (undefined)

   $\sec 360° = 1/1 = 1$

   $\csc 360° = 1/0$ (undefined)

3. $\sin 585° = -1/\sqrt{2} = -\sqrt{2}/2$

$\cos 585° = -1/\sqrt{2} = -\sqrt{2}/2$

$\tan 585° = -1/-1 = 1$

$\cot 585° = -1/-1 = 1$

$\sec 585° = \sqrt{2}/-1 = -\sqrt{2}$

$\csc 585° = \sqrt{2}/-1 = -\sqrt{2}$

4. $\sin(-180°) = 0/1 = 0$

$\cos(-180°) = -1/1 = -1$

$\tan(-180°) = 0/-1 = 0$

$\cot(-180°) = -1/0$ (undefined)

$\sec(-180°) = 1/-1 = -1$

$\csc(-180°) = 1/0$ (undefined)

5. $\sin(-315°) = 1/\sqrt{2} = \sqrt{2}/2$

$\cos(-315°) = 1/\sqrt{2} = \sqrt{2}/2$

$\tan(-315°) = 1/1 = 1$

$\cot(-315°) = 1/1 = 1$

$\sec(-315°) = \sqrt{2}/1 = \sqrt{2}$

$\csc(-315°) = \sqrt{2}/1 = \sqrt{2}$

6. $(1/2)^2 + (-\sqrt{3}/2)^2 = 1$

7. $2(\sqrt{3}/2)(-1/2) = -\sqrt{3}/2$

8. $(-1/\sqrt{2})^2 - (1/\sqrt{2})^2 = 0$

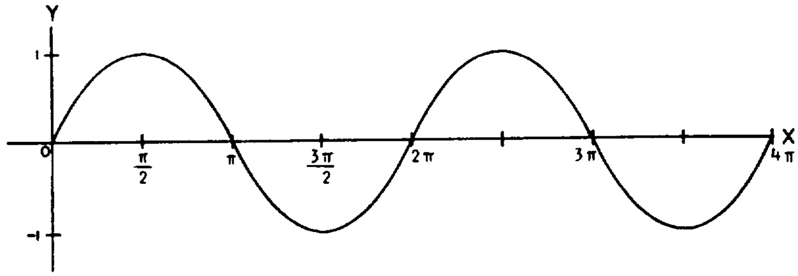


Figure 4-22.—Graph of the sine function.

PERIODS OF THE TRIGONOMETRIC FUNCTIONS

A trigonometric function of an angle is not changed in value when the angle is changed by any multiple of 360° or 2π radians. For this reason the functions are said to be *periodic.*

In the following paragraphs, the graphs of the sine, cosine, and tangent functions are discussed.

GRAPH OF THE SINE FUNCTION

Figure 4-22 shows two periods of the sine function. The graph shows that the value of the sine function varies between $+1$ and -1 and never goes beyond these limits as the angle varies. The graph also shows that the sine function increases from 0 at 0° or 0 radians to a maximum value of $+1$ at 90° or $\pi/2$ radians. It decreases back to 0 at 180° or π radians and continues to decrease to a minimum value of -1 at 270° or $3\pi/2$ radians. It then increases to a value of 0 at 360° or 2π radians. If we extend the graph (in either direction), the sine function will continue to repeat itself every 360° or 2π radians. Therefore, *the period of the sine function is 360° or 2π radians.*

GRAPH OF THE COSINE FUNCTION

The cosine function also has a period of 360° or 2π radians. Figure 4-23 shows two periods of the cosine function. The range

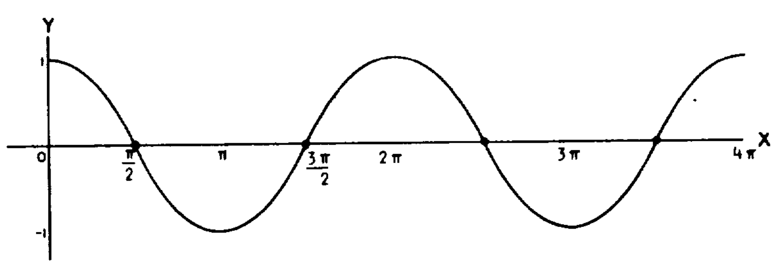


Figure 4-23.—Graph of the cosine function.

of the values the cosine function takes on also lies between +1 and −1. However, as seen on the graph, the cosine function decreases from 1 at 0° or 0 radians to 0 at 90° or $\pi/2$ radians and continues to decrease to a minimum value of −1 at 180° or π radians. It then increases to 0 at 270° or $3\pi/2$ radians and continues to increase to a maximum value of +1 at 360° or 2π radians. This completes one period of the cosine function.

GRAPH OF THE TANGENT FUNCTION

Figure 4-24 shows the graph of the tangent function from 0 radians to 2π radians. Notice that the tangent function is 0 at 0° or 0 radians and increases to positive infinity (without bounds) between 0° and 90° or 0 radians and $\pi/2$ radians. Remember that the tangent function is undefined for $90° + n(180°)$ or $\pi/2 + n\pi$, where n is any integer. The dashed vertical lines in figure 4-24 represent the undefined points. The tangent function increases from negative infinity to 0 between 90° and 180° or $\pi/2$ radians and π radians. At 180° or π radians, the tangent function is 0. The function continues to increase from 0 to positive infinity between 180° and 270° or π radians and $3\pi/2$ radians. Between 270° and 360° or $3\pi/2$ radians and 2π, it again increases from negative infinity to 0 at 360° or 2π radians. If we extend the graph (in either direction), the curve will repeat itself every 180° or π radians. Therefore, *the period of the tangent function is 180° or π radians.*

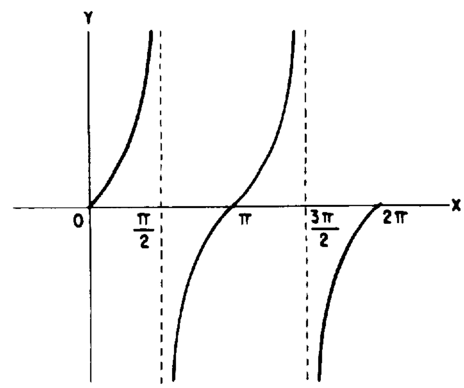


Figure 4-24.—Graph of the tangent function.

EXAMPLE: Using the graphs in figures 4-22 through 4-24, determine the values of θ, where $\sin\theta$ and $\tan\theta$ increase together, if $0 \leq \theta \leq \pi$.

SOLUTION: In figure 4-22 the sine function increases between 0 and $\pi/2$ radians for the interval of $0 \leq \theta \leq \pi$. (The sine function does not increase or decrease at points 0 or $\pi/2$.) In figure 4-24 the tangent function also increases between 0 and $\pi/2$ radians for the interval of $0 \leq \theta \leq \pi$. (The tangent function does not increase or decrease at 0 and is undefined at $\pi/2$.) Therefore, the values of θ, where $\sin\theta$ and $\tan\theta$ increase together, are $0 < \theta < \pi/2$.

PRACTICE PROBLEMS:

Use the graphs in figures 4-22 through 4-24 to answer the following problems (use appendixes II and III to verify your answers):

1. For what values of θ does $\cos \theta$ increase if $0 \leq \theta \leq \pi$?

2. For what values of θ do $\sin \theta$ and $\cos \theta$ decrease together if $0 \leq \theta \leq 2\pi$?

3. For what values of θ do $\cos \theta$ and $\tan \theta$ increase together if $\pi/2 \leq \theta \leq 3\pi/2$?

4. For what values of θ do $\sin \theta$, $\cos \theta$, and $\tan \theta$ increase together if $0 \leq \theta \leq 2\pi$?

ANSWERS:

1. None

2. $\pi/2 < \theta < \pi$

3. $\pi < \theta < 3\pi/2$

4. $3\pi/2 < \theta < 2\pi$

SUMMARY

The following are the major topics covered in this chapter:

1. **Angles in standard position:** An angle in standard position on a rectangular coordinate system has its vertex at the origin, its initial side lying along the X axis, and its terminal side lying in any of the quadrants or on one of the axes.

2. **Coterminal angles**: When two or more angles in standard position have their terminal sides located at the same position, they are said to be *coterminal.*

   For any general angle θ measured in degrees, any angle ϕ coterminal with θ can be found by

   $$\phi = \theta + n(360°)$$

   where n is any integer.

3. **Definitions of the trigonometric functions:**

$$\sin \theta = \frac{y}{r} = \frac{\text{ordinate}}{\text{length of radius}}$$

$$\cos \theta = \frac{x}{r} = \frac{\text{abscissa}}{\text{length of radius}}$$

$$\tan \theta = \frac{y}{x} = \frac{\text{ordinate}}{\text{abscissa}}$$

$$\cot \theta = \frac{x}{y} = \frac{\text{abscissa}}{\text{ordinate}}$$

$$\sec \theta = \frac{r}{x} = \frac{\text{length of radius}}{\text{abscissa}}$$

$$\csc \theta = \frac{r}{y} = \frac{\text{length of radius}}{\text{ordinate}}$$

4. **Signs of the trigonometric ratios in the quadrant system**: All the trigonometric ratios are positive for angles in the first quadrant. Only the sine and cosecant ratios are positive in the second quadrant. Only the tangent and cotangent ratios are positive in the third quadrant. Only the cosine and secant ratios are positive in the fourth quadrant.

5. **Reference angle**: The *reference angle*, θ', for any angle, θ, in standard position is the smallest positive angle between the radius vector of θ and the X axis, such that $0° \leq \theta' \leq 90°$. In general, for any integer n,

$$\theta' = n(180°) \pm \theta$$

or

$$\theta' = n\pi \pm \theta$$

where $0 \leq \theta' \leq \pi/2$.

6. **Reference triangle:** The right triangle formed from the reference angle when you connect a point on the radius vector of the reference angle perpendicular to the X axis is called the *reference triangle*.

7. **Reduction formulas:** *Reduction formulas* are formulas used to determine the signs of the functions of any angle. They provide a means of reducing the functions of any angle to an equivalent expression for the function in terms of a positive acute angle.

8. **Quadrant I angles**: An angle in quadrant I is represented by θ.

$$\sin\theta = y/r$$

$$\cos\theta = x/r$$

$$\tan\theta = y/x$$

$$\cot\theta = x/y$$

$$\sec\theta = r/x$$

$$\csc\theta = r/y$$

9. **Quadrant II angles**: An angle in quadrant II is represented by $180 - \theta$.

$$\sin(180° - \theta) = \sin\theta$$

$$\cos(180° - \theta) = -\cos\theta$$

$$\tan(180° - \theta) = -\tan\theta$$

$$\cot(180° - \theta) = -\cot\theta$$

$$\sec(180° - \theta) = -\sec\theta$$

$$\csc(180° - \theta) = \csc\theta$$

10. **Quadrant III angles**: An angle in quadrant III is represented by $180° + \theta$.

$$\sin(180° + \theta) = -\sin\theta$$

$$\cos(180° + \theta) = -\cos\theta$$

$$\tan(180° + \theta) = \tan\theta$$

$$\cot(180° + \theta) = \cot\theta$$

$$\sec(180° + \theta) = -\sec\theta$$

$$\csc(180° + \theta) = -\csc\theta$$

11. **Quadrant IV angles:** An angle in quadrant IV is represented by $360° - \theta$.

$$\sin(360° - \theta) = -\sin\theta$$

$$\cos(360° - \theta) = \cos\theta$$

$$\tan(360° - \theta) = -\tan\theta$$

$$\cot(360° - \theta) = -\cot\theta$$

$$\sec(360° - \theta) = \sec\theta$$

$$\csc(360° - \theta) = -\csc\theta$$

12. **Functions of negative angles:**

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

$$\cot(-\theta) = -\cot\theta$$

$$\sec(-\theta) = \sec\theta$$

$$\csc(-\theta) = -\csc\theta$$

13. **Functions of coterminal angles**: For a coterminal angle in the form of

$$\theta' = n(360°) + \theta$$

where n is any integer and θ is an integral multiple of θ', the trigonometric functions of θ' are equal to those of θ.

14. **Cofunctions and complementary angles**: *Complementary angles* are angles whose sum is 90°. Two trigonometric functions that have equal values for complementary angles are called *cofunctions*.

$$\sin\theta = \cos(90° - \theta)$$

$$\cos\theta = \sin(90° - \theta)$$

$$\tan\theta = \cot(90° - \theta)$$

$$\cot\theta = \tan(90° - \theta)$$

$$\sec\theta = \csc(90° - \theta)$$

$$\csc\theta = \sec(90° - \theta)$$

15. **Frequently used angles:** The trigonometric functions of 30°, 60°, and 45° can be determined geometrically. The trigonometric ratios corresponding to these functions are summarized in table 4-1.

16. **Quadrantal angles**: An angle whose terminal side lies on a coordinate axis when the angle is in standard position is a *quadrantal angle*. The trigonometric ratios corresponding to the functions of the quadrantal angles are summarized in table 4-2.

17. **Periods of the trigonometric functions:** A trigonometric function of an angle is not changed in value when the angle is changed by any multiple of 360° or 2π radians. For this reason the functions are said to be *periodic*. The periods of the sine and cosine functions are 360° or 2π radians. The period of the tangent function is 180° or π radians.

ADDITIONAL PRACTICE PROBLEMS

1. Are the angles $840°$, $-240°$, and $600°$ coterminal?

2. Find the sine, cosine, and tangent of the angle θ whose radius vector passes through the point $P(\sqrt{5}, \sqrt{11})$.

3. Find the six trigonometric functions of θ if $\csc \theta = -37/35$ and $\tan \theta > 0$.

4. Find the sine, cosine, and tangent of $-4{,}010°$.

5. Express csc $87° \, 23' \, 13''$ as a function of its complementary angle.

6. Without using the appendixes, evaluate $\sec^2(-135°) + \cot^2(-690°) + \csc^2(-600°)$.

7. Without using the appendixes, find the six trigonometric functions of $-3{,}510°$.

8. For what values of θ do $\cos \theta$ and $\tan \theta$ both increase and $\sin \theta$ decrease together if $0 \leq \theta \leq 2\pi$?

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. No

2. $\sin \theta = \sqrt{11}/4$

   $\cos \theta = \sqrt{5}/4$

   $\tan \theta = \sqrt{11}/\sqrt{5} = \sqrt{55}/5$

3. $\sin \theta = -35/37$

   $\cos \theta = -12/37$

   $\tan \theta = 35/12$

   $\cot \theta = 12/35$

   $\sec \theta = -37/12$

   $\csc \theta = -37/35$

4. $\sin \theta = -0.76604$

   $\cos \theta = 0.64279$

   $\tan \theta = -1.19175$

5. $\sec 2° 36' 47''$

6. 6 1/3

7. $\sin \theta = 1$

   $\cos \theta = 0$

   $\tan \theta$ is undefined

   $\cot \theta = 0$

   $\sec \theta$ is undefined

   $\csc \theta = 1$

8. $\pi < \theta < 3\pi/2$

CHAPTER 5

OBLIQUE TRIANGLES

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Apply the Law of Sines to solve oblique triangles given one side and two angles or two sides and an angle opposite one of them.

2. Apply the Law of Cosines to solve oblique triangles given two sides and the included angle or all three sides.

3. Find the area of an oblique triangle.

INTRODUCTION

The two previous chapters primarily dealt with properties of right triangles in solving trigonometric measurements and functions. In this chapter we will apply properties of oblique triangles in solving trigonometric measurements and functions. *Oblique triangles* are triangles containing no right angles. Oblique triangles are made up of either three acute angles or two acute angles and one obtuse angle. *Acute angles* have measures between $0°$ and $90°$. *Obtuse angles* have measures between $90°$ and $180°$.

In *Mathematics,* Volume 1, a method for solving problems involving oblique triangles was introduced. The method employed the procedures of dividing the original triangle into two or more right triangles and using the properties of right triangles in problem solving.

This chapter develops two methods or laws dealing directly with oblique triangles. The methods consider the parts of the triangle that are given. The four standard cases for solving oblique triangles are as follows:

Case 1. One side and two angles

Case 2. Two sides and an angle opposite one of them

Case 3. Two sides and the included angle

Case 4. All three sides

Also included in this chapter are problems concerning the area of a triangle, which combine the area formula of plane geometry with trigonometric properties.

METHODS OF SOLVING OBLIQUE TRIANGLES

This section is concerned with the development and proofs of the Law of Sines and the Law of Cosines. The four standard cases for solving oblique triangles use applications of these laws.

LAW OF SINES

Law of Sines. *The lengths of the sides of any triangle are proportional to the sines of their opposite angles*; that is,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

PROOF: Refer to the oblique triangle shown in figure 5-1, view A. Let h be the length of the perpendicular from angle A to the side opposite angle A. Considering the two right triangles formed by h, we obtain

$$\sin B = \frac{h}{c} \text{ or } h = c \sin B$$

and

$$\sin C = \frac{h}{b} \text{ or } h = b \sin C$$

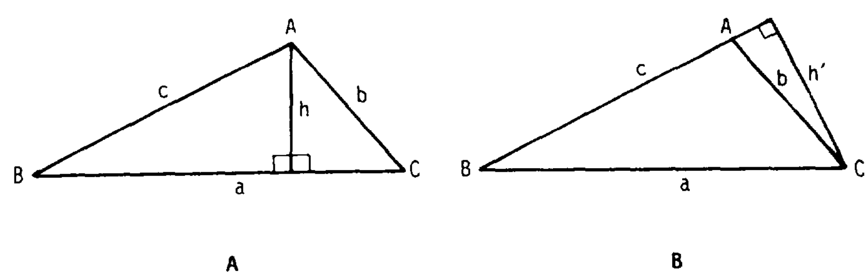


Figure 5-1.—Development of Law of Sines.

Equating these two values of h, we have

$$c \sin B = b \sin C$$

or in an equivalent form, we have

$$\frac{c}{\sin C} = \frac{b}{\sin B}$$

Now, if we redraw the oblique triangle in figure 5-1, view A, by extending the length of side c until it forms a right angle (is perpendicular) with a line, h', from angle C (see fig. 5-1, view B), then from the newly formed triangle, we obtain

$$\sin B = \frac{h'}{a} \text{ or } h' = a \sin B$$

and

$$\sin (180° - A) = \frac{h'}{b} \text{ or } h' = b \sin (180° - A)$$

Since

$$\sin (180° - A) = \sin A$$

then by substituting $\sin A$ for $\sin (180° - A)$ and equating values of h', we have

$$a \sin B = b \sin A$$

or in an equivalent form

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

But

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

Therefore,

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

Case 1. One Side and Two Angles

When one side and two angles of a triangle are given, the third angle can be found since the sum of the angles equals 180°; that is, $A + B + C = 180°$. Then the Law of Sines can be used to find the two remaining sides.

EXAMPLE: Solve the remaining parts of triangle ABC, given $c = 5$, $B = 30°$, and $C = 97° 30'$. Give side accuracy to one decimal place.

SOLUTION: Refer to figure 5-2. Since

$$A + B + C = 180°$$

then

$$A + 30° + 97° 30' = 180°$$

$$A = 180° - 30° - 97° 30'$$

$$= 52° 30'$$

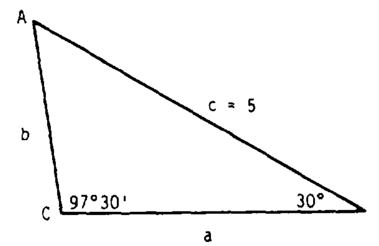


Figure 5-2.—Case 1. One side and two angles.

By the Law of Sines,

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

we obtain

$$\frac{a}{\sin 52° 30'} = \frac{5}{\sin 97° 30'}$$

$$a = \frac{5 \sin 52° 30'}{\sin 97° 30'}$$

$$= \frac{5(0.79335)}{0.99144}$$

$$= 4.0$$

We will use the Law of Sines again to solve for the length of side b:

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

so,

$$\frac{b}{\sin 30^\circ} = \frac{5}{\sin 97^\circ 30'}$$

$$b = \frac{5 \sin 30^\circ}{\sin 97^\circ 30'}$$

$$= \frac{5(0.50000)}{0.99144}$$

$$= 2.5$$

EXAMPLE: The base of flagpole standing vertically on a hill is inclined at an angle of 15° with the horizontal. A man standing 200 feet downhill from the base of the flagpole notes that his line of sight to the top of the flagpole makes an angle of 40° with the horizontal. How high, to the nearest foot, is the flagpole?

SOLUTION: Refer to figure 5-3. In triangle *ABC* we find

$$A = 40^\circ - 15^\circ$$

$$= 25^\circ$$

From right triangle *ADC* we find

$$C = 180^\circ - 40^\circ - 90^\circ$$

$$= 50^\circ$$

Applying the Law of Sines, we obtain

$$\frac{a}{\sin 25^\circ} = \frac{200}{\sin 50^\circ}$$

$$a = \frac{200 \sin 25^\circ}{\sin 50^\circ}$$

$$= \frac{200(0.42262)}{0.76604}$$

$$= 110 \text{ feet}$$

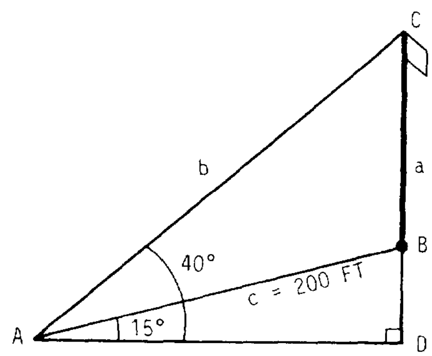


Figure 5-3.—Case 1. Flagpole problem.

Case 2. Two Sides and an Angle Opposite One of Them

Case 2 is sometimes referred to as the *ambiguous case* since two triangles, one triangle, or no triangle

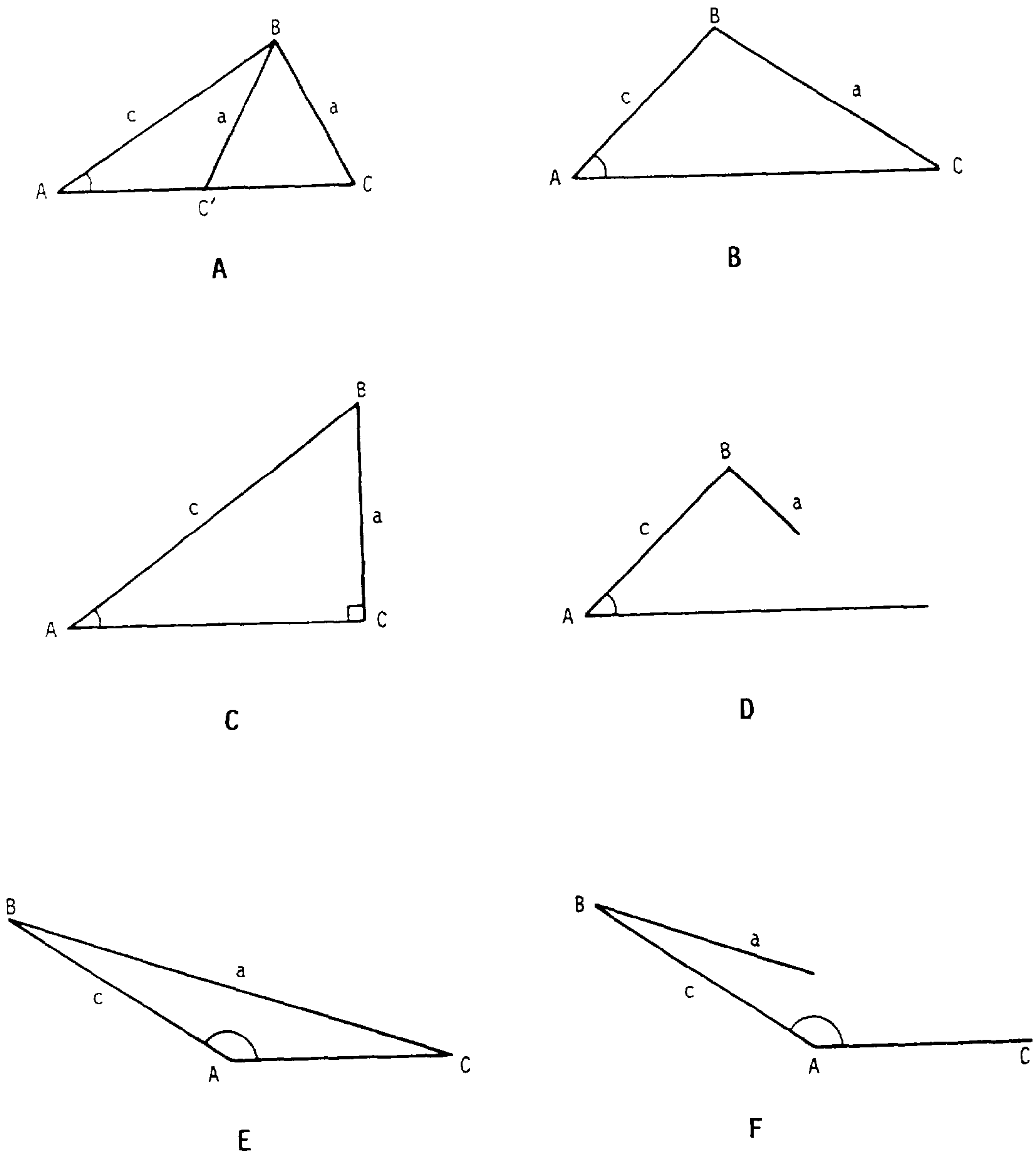


Figure 5-4.—Case 2. Two sides and an angle opposite one of them.

may result from data given in this form. Consider triangle *ABC* in figure 5-4. Assuming we are given angle *A* and sides *a* and *c*, the following situations may exist:

For acute angle *A*:

1. If $a < c$ and $\sin C < 1$, then two possible triangles exist; one triangle comprises the acute angle *C* and the other triangle comprises the obtuse angle $C' = 180° - C$. See figure 5-4, view A.

2. If $a \geq c$, then one triangle exists. See figure 5-4, view *B*.

3. If $a < c$ and $\sin C = 1$, then one right triangle exists. See figure 5-4, view C.

4. If $a < c$ and $\sin C > 1$, then no triangle is determined. See figure 5-4, view D. (This should be obvious since in the previous chapter we learned that the sine of an angle may have values only between 0 and 1.)

For obtuse angle A:

1. If $a > c$, then one triangle exists. See figure 5-4, view E.

2. If $a \leq c$, then no triangle is determined. See figure 5-4, view F.

When two sides and an angle opposite one of them are given, we can solve for the remaining parts of the triangle using the Law of Sines. Sketches can be helpful.

EXAMPLE: Solve the triangle or triangles if they exist, given $B = 45°$, $b = 3$, and $c = 7$.

SOLUTION: Using the Law of Sines,

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

we have

$$\frac{3}{\sin 45°} = \frac{7}{\sin C}$$

$$\sin C = \frac{7 \sin 45°}{3}$$

$$= \frac{7(0.70711)}{3}$$

$$= 1.64992$$

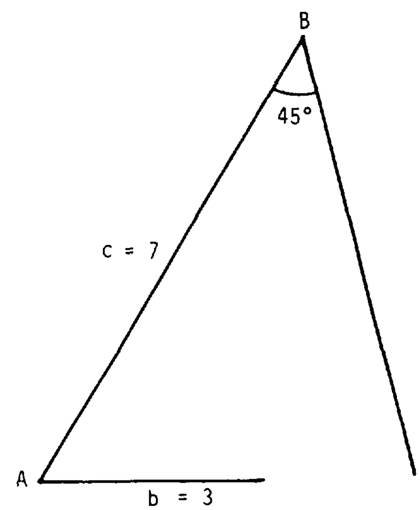


Figure 5-5.—Case 2. Acute angle *A* with $b < c$ and $\sin C > 1$.

Since the sine of an angle cannot exceed 1, then we conclude that no triangle exists. Refer to figure 5-5. Notice that $b < c$ and $\sin C > 1$.

EXAMPLE: Solve the triangle or triangles if they exist, given $A = 22°$, $a = 5.4$, and $c = 14$. Give angle accuracy to the nearest minute and side accuracy to one decimal place.

SOLUTION: Using the Law of Sines,

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

we obtain

$$\frac{5.4}{\sin 22°} = \frac{14}{\sin C}$$

$$\sin C = \frac{14 \sin 22°}{5.4}$$

$$= \frac{14(0.37461)}{5.4}$$

$$= 0.97121$$

$$C = 76° \ 13'$$

Since the side opposite the known angle is smaller than the other given side, that is, $a < c$, and $\sin C < 1$, then two possible triangles exist. One triangle is ABC and the other is $AB'C'$. Refer to figure 5-6. Hence,

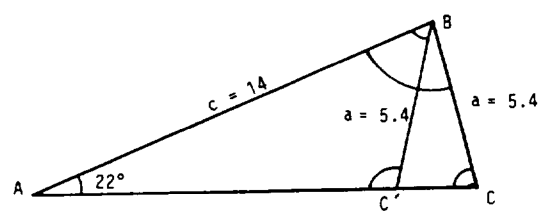


Figure 5-6.—Case 2. Acute angle A with $a < c$ and $\sin C < 1$.

$$C' = 180° - C$$

$$= 180° - 76° \ 13'$$

$$= 103° \ 47'$$

Solving triangle ABC first, we find angle B by

$$B = 180° - (A + C)$$

$$= 180° - (22° + 76° \ 13')$$

$$= 81° \ 47'$$

and by the Law of Sines,

$$\frac{a}{\sin A} = \frac{b}{\sin B}$$

we find the length of side b to be

$$\frac{5.4}{\sin 22^\circ} = \frac{b}{\sin 81^\circ 47'}$$

$$b = \frac{5.4 \sin 81^\circ 47'}{\sin 22^\circ}$$

$$= \frac{5.4(0.98973)}{0.37461}$$

$$= 14.3$$

Now solving triangle $AB'C'$, we find angle B' by

$$B' = 180^\circ - (A + C')$$

$$= 180^\circ - (22^\circ + 103^\circ 47')$$

$$= 54^\circ 13'$$

and by the Law of Sines,

$$\frac{a}{\sin A} = \frac{b'}{\sin B'}$$

we find the length of side b' to be

$$\frac{5.4}{\sin 22^\circ} = \frac{b'}{\sin 54^\circ 13'}$$

$$b' = \frac{5.4 \sin 54^\circ 13'}{\sin 22^\circ}$$

$$= \frac{5.4(0.81123)}{0.37461}$$

$$= 11.7$$

EXAMPLE: Solve the triangle if it exists, given $C = 125^\circ 48'$, $b = 41.8$, and $c = 56.2$. Give angle accuracy to the nearest minute and side accuracy to two decimal places.

SOLUTION: Since the given angle, C, is obtuse and the side opposite the given angle is larger than the other given side, that is, $c > b$, then one triangle exists. Refer to figure 5-7. By the Law of Sines,

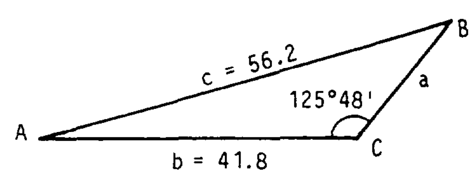


Figure 5-7.—Case 2. Obtuse angle A with $c > b$.

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

we get

$$\frac{41.8}{\sin B} = \frac{56.2}{\sin 125^\circ 48'}$$

$$\sin B = \frac{41.8 \sin 125^\circ 48'}{56.2}$$

$$= \frac{41.8(0.81106)}{56.2}$$

$$= 0.60324$$

$$B = 37^\circ 6'$$

Additionally,

$$A = 180^\circ - (B + C)$$

$$= 180^\circ - (37^\circ 6' + 125^\circ 48')$$

$$= 17^\circ 6'$$

and by the Law of Sines,

$$\frac{a}{\sin A} = \frac{c}{\sin C}$$

we find the length of side a to be

$$\frac{a}{\sin 17^\circ 6'} = \frac{56.2}{\sin 125^\circ 48'}$$

$$a = \frac{56.2 \sin 17^\circ 6'}{\sin 125^\circ 48'}$$

$$= \frac{56.2(0.29404)}{0.81106}$$

$$= 20.37$$

PRACTICE PROBLEMS:

Use the Law of Sines to solve the remaining parts of triangle ABC given the following parts (give angle accuracy to the nearest minute and side accuracy to one decimal place):

1. $A = 59° 36'$, $B = 48° 14'$, and $c = 86.4$
2. $A = 98° 8'$, $C = 25° 25'$, and $b = 2.1$
3. $B = 30° 30'$, $a = 10$, and $b = 10$
4. $C = 100° 21'$, $a = 4.2$, and $c = 3.2$

ANSWERS:

1. $C = 72° 10'$

   $a = 78.3$

   $b = 67.7$

2. $B = 56° 27'$

   $a = 2.5$

   $c = 1.1$

3. $A = 30° 30'$

   $C = 119°$

   $c = 17.2$

4. No solution; C is obtuse and $c \leq a$.

LAW OF COSINES

Law of Cosines. *In a triangle, the square of any side is equal to the sum of the squares of the other two sides minus twice the* *product of the same two sides multiplied by the cosine of the angle between them*; that is,

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

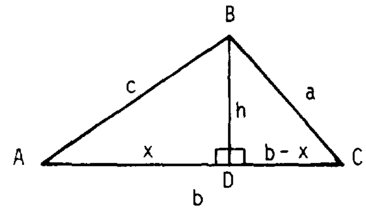


Figure 5-8.—Development of Law of Cosines.

PROOF: Refer to the oblique triangle shown in figure 5-8. Let h be the length of the perpendicular from angle B to the side opposite angle B.

NOTE: $b = b + 0$

$$= b + (x - x)$$

$$= x + (b - x)$$

Considering right triangle ADB formed by h, we obtain

$$\cos A = \frac{x}{c} \text{ or } x = c \cos A$$

and

$$h^2 = c^2 - x^2$$

Substituting the value of x into the last equation gives

$$h^2 = c^2 - c^2 \cos^2 A$$

Considering right triangle CDB, we obtain

$$h^2 = a^2 - (b - x)^2$$

$$= a^2 - b^2 + 2bx - x^2$$

Substituting the value x in the last equation for h^2 gives

$$h^2 = a^2 - b^2 + 2bc \cos A - c^2 \cos^2 A$$

Equating the two values of h^2 gives

$$c^2 - c^2 \cos^2 A = a^2 - b^2 + 2bc \cos A - c^2 \cos^2 A$$

Therefore, rearranging and canceling terms gives

$$a^2 = b^2 + c^2 - 2bc \cos A$$

The same procedure can be applied to derive all three forms of the Law of Cosines, which are

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

Case 3. Two Sides and the Included Angle

When two sides and the angle between them are given, we can solve for the remaining parts of the triangle using the Law of Cosines. First, the unknown side is determined; then the two other angles are determined.

EXAMPLE: Solve for the remaining parts of triangle ABC, given $b = 7$, $c = 5$, and $A = 19°$. Give angle accuracy to the nearest minute and side accuracy to one decimal place.

SOLUTION: Refer to figure 5-9. First, find the length of the unknown side using the Law of Cosines,

$$a^2 = b^2 + c^2 - 2bc \cos A$$

Hence,

$$a^2 = 7^2 + 5^2 - 2(7)(5) \cos 19°$$

$$= 49 + 25 - 70(0.94552)$$

$$= 7.8136$$

$$a = \sqrt{7.8136}$$

$$= 2.8$$

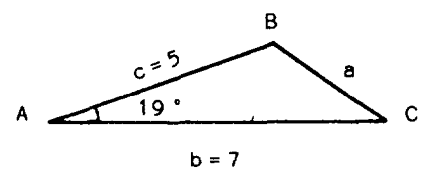


Figure 5-9.—Case 3. Two sides and the included angle.

Next, compute the remaining angles using a rearrangement of the Law of Cosines:

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{(2.8)^2 + (5)^2 - (7)^2}{2(2.8)(5)}$$

$$= \frac{7.84 + 25 - 49}{28}$$

$$= -0.57714$$

$$= -\cos 54° 45'$$

Angle B is an obtuse angle since $\cos B$ is negative. Therefore,

$$B = 125° 15'$$

and

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{(2.8)^2 + (7)^2 - (5)^2}{2(2.8)(7)}$$

$$= \frac{7.84 + 49 - 25}{39.2}$$

$$= 0.81224$$

$$C = 35° 41'$$

NOTE: Since we are solving angles to the nearest minute, the sum of the angles may not equal exactly 180°.

EXAMPLE: Two points, A and B, are separated by a pond. The distance from A to a third point, C, is 10.2 feet; the distance from C to B is 13.8 feet; and angle C is 52° 40′. Find the distance from A to B to two decimal places and angles A and B to the nearest minute.

SOLUTION: We will first find the distance from point A to point B. Using the Law of Cosines, we find that

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$= (13.8)^2 + (10.2)^2 - 2(13.8)(10.2) \cos 52° 40'$$

$$= 190.44 + 104.04 - (281.52)(0.60645)$$

$$= 123.75$$

$$c = \sqrt{123.75}$$

$$= 11.12 \text{ feet}$$

Now we will find angles A and B using the Law of Cosines:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(10.2)^2 + (11.12)^2 - (13.8)^2}{2(10.2)(11.12)}$$

$$= 0.16423$$

$$A = 80° 33'$$

and

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{(13.8)^2 + (11.12)^2 - (10.2)^2}{2(13.8)(11.12)}$$

$$= 0.68441$$

$$B = 46° \ 49'$$

Case 4. All Three Sides

The Law of Cosines can also be used to find the size of the angles of a triangle when the length of all three sides are given.

EXAMPLE: Find the measure of each angle (to the nearest minute) of a triangle having sides $a = 7$, $b = 13$, and $c = 14$.

SOLUTION:

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(13)^2 + (14)^2 - (7)^2}{2(13)(14)}$$

$$= 0.86813$$

$$A = 29° \ 45'$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$= \frac{(7)^2 + (14)^2 - (13)^2}{2(7)(14)}$$

$$= 0.38776$$

$$B = 67° \ 11'$$

and

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$

$$= \frac{(7)^2 + (13)^2 - (14)^2}{2(7)(13)}$$

$$= 0.12088$$

$$C = 83° \ 3'$$

EXAMPLE: A triangular plot of ground measures 50 meters by 70 meters by 90 meters. Find, to the nearest minute, the size of the angle, A, opposite the longest side.

SOLUTION:

$$\cos A = \frac{(50)^2 + (70)^2 - (90)^2}{2(50)(70)}$$

$$= -0.10000$$

$$= -\cos 84° \, 16'$$

$$A = 95° \, 44'$$

PRACTICE PROBLEMS:

Use the Law of Cosines to solve the remaining parts of triangle ABC given the following parts (give angle accuracy to the nearest minute and side accuracy to two decimal places):

1. $a = 54.2$, $c = 83.4$, and $B = 111° \, 11'$
2. $b = 6.6$, $c = 6.6$, and $A = 60°$
3. $a = 22.2$, $b = 33.3$, and $c = 44.4$
4. $a = 15.6$, $b = 16.7$, and $c = 17.8$

ANSWERS:

1. $b = 114.72$
   $A = 26° \, 8'$
   $C = 42° \, 41'$

2. $a = 6.6$
   $B = 60°$
   $C = 60°$

3. $A = 28° \, 57'$
   $B = 46° \, 34'$
   $C = 104° \, 29'$

4. $A = 53° \, 39'$
   $B = 59° \, 34'$
   $C = 66° \, 47'$

AREA FORMULAS

In this section two formulas for finding the area of a triangle will be developed. Recall from plane geometry that the area of a triangle is found by the formula

$$\text{area} = \frac{1}{2}bh$$

where b is any side of the triangle and h is the altitude drawn to that side. While this is a useful formula, it is not a practical one. With the help of trigonometry, we can derive more practical formulas for the area of a triangle.

Consider the triangle in figure 5-8. The length of the altitude is found to be

$$h = c \sin A$$

Substituting this value of h into the geometric area formula results in

$$\text{area} = \frac{1}{2}b(c \sin A)$$

$$= \frac{1}{2}bc \sin A$$

In general, *the area of a triangle is equal to one-half the product of the lengths of any two sides and the sine of their included angle;* that is,

$$\text{area} = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B = \frac{1}{2}bc \sin A$$

EXAMPLE: Find the area of triangle ABC to one decimal place if $a = 13$, $b = 9$, and $C = 40°$.

SOLUTION: Since C is the angle between sides a and b, the area formula is

$$\text{area} = \frac{1}{2}ab \sin C$$

so

$$\text{area} = \frac{1}{2}(13)(9) \sin 40°$$

$$= 58.5(0.64279)$$

$$= 37.6$$

Another formula for the area of a triangle can be derived by the use of the Law of Sines and the previous formula. From the Law of Sines,

$$\frac{b}{\sin B} = \frac{c}{\sin C}$$

we find

$$b = \frac{c \sin B}{\sin C}$$

Substituting this value of b into the previous area formula

$$\text{area} = \frac{1}{2}bc \sin A$$

results in

$$\text{area} = \frac{1}{2}\left(\frac{c \sin B}{\sin C}\right)c \sin A$$

$$= \frac{c^2 \sin A \sin B}{2 \sin C}$$

Therefore, *the area of a triangle can be determined if one side and two angles are known* (since the third angle can be found directly); that is,

$$\text{area} = \frac{a^2 \sin B \sin C}{2 \sin A} = \frac{b^2 \sin A \sin C}{2 \sin B} = \frac{c^2 \sin A \sin B}{2 \sin C}$$

EXAMPLE: Find the area of triangle ABC to one decimal place if $A = 25°$, $C = 105°$, and $b = 12$.

SOLUTION: First, we find B to be

$$B = 180° - (A + C)$$

$$= 180° - (25° + 105°)$$

$$= 50°$$

The area formula for this situation would be

$$\text{area} = \frac{b^2 \sin A \sin C}{2 \sin B}$$

so,

$$\text{area} = \frac{(12)^2 \sin 25° \sin 105°}{2 \sin 50°}$$

$$= \frac{144(0.42262)(0.96593)}{2(0.76604)}$$

$$= 38.4$$

PRACTICE PROBLEMS:

Find the area of triangle ABC to three decimal places given the following measurements:

1. $b = 20.02$, $c = 40.04$, and $A = 80° 8'$

2. $a = 3.28$, $c = 9.18$, and $B = 42° 21'$

3. $B = 50°$, $C = 70°$, and $c = 5.07$

4. $A = 103° 48'$, $B = 34° 6'$, and $a = 4.24$

ANSWERS:

1. 394.873

2. 10.142

3. 9.074

4. 3.479

SUMMARY

The following are the major topics covered in this chapter:

1. **Oblique triangles:** *Oblique triangles* are triangles containing no right angles. Oblique triangles are made up of either three acute angles or two acute angles and one obtuse angle.

   Acute angles have measures between 0° and 90°.

   Obtuse angles have measures between 90° and 180°.

2. **Law of Sines:** The lengths of the sides of any triangle are proportional to the sines of their opposite angles.

$$\frac{a}{\sin A} = \frac{b}{\sin B} \quad \frac{c}{\sin C}$$

3. **Standard cases for solving oblique triangles using the Law of Sines:**

   Case 1. One side and two angles

   Case 2. Two sides and an angle opposite one of them (This is referred to as the *ambiguous case* since two triangles, one triangle, or no triangle may result from the given data.)

4. **Law of Cosines:** In a triangle, the square of any side is equal to the sum of the squares of the other two sides minus twice the product of the same two sides multiplied by the cosine of the angle between them.

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

5. **Standard cases for solving oblique triangles using the Law of Cosines:**

   Case 3. Two sides and the included angle

   Case 4. All three sides

6. **Area of a triangle:**

The area of a triangle is equal to one-half the product of the lengths of any two sides and the sine of their included angle.

$$\text{area} = \frac{1}{2}ab \sin C = \frac{1}{2}ac \sin B = \frac{1}{2}bc \sin A$$

The area of a triangle can be determined if one side and two angles are known.

$$\text{area} = \frac{a^2 \sin B \sin C}{2 \sin A} = \frac{b^2 \sin A \sin C}{2 \sin B} = \frac{c^2 \sin A \sin B}{2 \sin C}$$

ADDITIONAL PRACTICE PROBLEMS

Use the Law of Sines, Law of Cosines, or area formulas to solve the following problems:

1. To determine the distance from point A to point B across a canyon, Barbara lays off a distance from point C to point B as 440 yards. She then finds that $C = 30° 17'$ and $B = 104° 53'$. What is the distance, to the nearest yard, between points A and B?

2. Two buoys are 325 feet apart and a boat is 250 feet from one of them. The angle subtended by the two buoys at the boat is $65° 10'$. Find the distance, to the nearest foot, from the boat to the other buoy.

3. A triangular tract of land is to be enclosed by a fence. Side a equals 37.25 feet, side c equals 46.98 feet, and the included angle B is $100° 30'$. Find the amount of fencing, to the nearest hundredth of a foot, needed to enclose the triangular plot.

4. A 12-foot ladder is placed against an inclined support and reaches 10 feet up the side of the support. The foot of the ladder is 5 feet from the foot of the inclined support. What is the measure of the angle, to the nearest minute, the ladder makes with the support?

5. Find the area, to one decimal place, of a triangular field if two sides of the field are 127 yards and 159 yards and the included angle is $57° 18'$.

6. What is the area of a parallelogram, to one decimal place, if the length of one diagonal is 6 inches and the diagonal meets two adjacent sides of the parallelogram at angles with measures $33°$ and $44°$? HINT: Double the area of a triangle.

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. 315 yards
2. 338 feet
3. 149.29 feet
4. 24° 9′
5. 8,496.3 square yards
6. 14.0 square inches

CHAPTER 6

TRIGONOMETRIC IDENTITIES AND EQUATIONS

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Apply the reciprocal, quotient, and Pythagorean identities along with identities for negative angles to problem solving.

2. Apply the sum and difference, double-angle, and half-angle formulas to problem solving.

3. Apply inverse trigonometric functions to problem solving.

4. Find solutions to trigonometric equations.

INTRODUCTION

This is the final chapter dealing directly with trigonometry and trigonometric relationships. This chapter includes the basic identities, formulas for identities involving more than one angle, and formulas for identities involving multiples of an angle.

Also included in this chapter are inverse trigonometric functions and methods for solving trigonometric equations.

FUNDAMENTAL IDENTITIES

An equality that is true for all values of an unknown is called an *identity*. Many of the identities that will be considered in this section were established in earlier chapters and will be used here to change the form of an expression.

Problems in identities are often given as equalities. The identity is established by either transforming the left side into the right side or transforming the right side into the left side. Never work across the equality sign.

We have no hard-and-fast rules to use in verifying identities. However, we do offer the following suggestions:

1. *Know the basic identities given in this section.*
2. *Attempt to transform the more complicated side into the other side.*
3. *When possible, express all trigonometric functions in the equation in terms of sine and cosine.*
4. *Perform any factoring or algebraic operations.*

RECIPROCAL IDENTITIES

The *reciprocal identities* were first introduced in chapter 3. They are as follows:

$$\cot \theta = \frac{1}{\tan \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

EXAMPLE: Use the reciprocal identities to find an equivalent expression involving only sines and cosines; then simplify for

$$\frac{\sec \theta}{\csc \theta + \sec \theta}$$

SOLUTION:

$$\frac{\sec \theta}{\csc \theta + \sec \theta} = \frac{\frac{1}{\cos \theta}}{\frac{1}{\sin \theta} + \frac{1}{\cos \theta}}$$

$$= \frac{\frac{1}{\cos \theta}}{\frac{\cos \theta + \sin \theta}{\sin \theta \cos \theta}}$$

$$= \left(\frac{1}{\cos \theta}\right)\left(\frac{\sin \theta \cos \theta}{\cos \theta + \sin \theta}\right)$$

$$= \frac{\sin \theta}{\cos \theta + \sin \theta}$$

QUOTIENT IDENTITIES

The *quotient identities* were also introduced in chapter 3.

They are as follows:

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

EXAMPLE: Use the quotient identities to find an equivalent expression involving only sines and cosines; then simplify for

$$\frac{\tan \theta}{\cot \theta}$$

SOLUTION:

$$\frac{\tan \theta}{\cot \theta} = \frac{\frac{\sin \theta}{\cos \theta}}{\frac{\cos \theta}{\sin \theta}}$$

$$= \left(\frac{\sin \theta}{\cos \theta}\right)\left(\frac{\sin \theta}{\cos \theta}\right)$$

$$= \frac{\sin^2\theta}{\cos^2\theta}$$

PYTHAGOREAN IDENTITIES

Another group of fundamental identities, called the *Pythagorean identities*, involves the squares of the functions. These identities are so named because the Pythagorean theorem is used in their development.

Consider

$$x^2 + y^2 = r^2$$

and divide both sides by r^2 to get

$$\frac{x^2}{r^2} + \frac{y^2}{r^2} = 1$$

or

$$\left(\frac{x}{r}\right)^2 + \left(\frac{y}{r}\right)^2 = 1$$

Since $\cos\theta = x/r$ and $\sin\theta = y/r$, then

$$(\cos\theta)^2 + (\sin\theta)^2 = 1$$

or

$$\cos^2\theta + \sin^2\theta = 1$$

which is one of the Pythagorean identities.

In the same manner, dividing both sides of the equation

$$x^2 + y^2 = r^2$$

by x^2 (where $x \neq 0$) gives

$$1 + \frac{y^2}{x^2} = \frac{r^2}{x^2}$$

or

$$1 + \left(\frac{y}{x}\right)^2 = \left(\frac{r}{x}\right)^2$$

Since $\tan\theta = y/x$ and $\sec\theta = r/x$, then

$$1 + (\tan\theta)^2 = (\sec\theta)^2$$

or

$$1 + \tan^2\theta = \sec^2\theta$$

which is another one of the Pythagorean identities.

Dividing both sides of the equation

$$x^2 + y^2 = r^2$$

by y^2 (where $y \neq 0$) gives

$$\frac{x^2}{y^2} + 1 = \frac{r^2}{y^2}$$

or

$$1 + \left(\frac{x}{y}\right)^2 = \left(\frac{r}{y}\right)^2$$

Since $\cot\theta = x/y$ and $\csc\theta = r/y$, then

$$1 + (\cot\theta)^2 = (\csc\theta)^2$$

or

$$1 + \cot^2\theta = \csc^2\theta$$

which is also one of the Pythagorean identities.

EXAMPLE: Use the Pythagorean identities to find an equivalent expression involving only sines and cosines; then simplify for

$$(\csc^2\theta - 1)(\tan^2\theta + 1)$$

SOLUTION:

$$\begin{aligned}(\csc^2\theta - 1)(\tan^2\theta + 1) &= \cot^2\theta \sec^2\theta \\ &= \left(\frac{\cos^2\theta}{\sin^2\theta}\right)\left(\frac{1}{\cos^2\theta}\right) \\ &= \frac{1}{\sin^2\theta}\end{aligned}$$

IDENTITIES FOR NEGATIVE ANGLES

The following *identities for negative angles* were first introduced in chapter 4:

$$\sin(-\theta) = -\sin\theta$$

$$\cos(-\theta) = \cos\theta$$

$$\tan(-\theta) = -\tan\theta$$

EXAMPLE: Use the identities for negative angles to find an equivalent expression involving only sines and cosines with positive angles; then simplify for

$$\frac{\tan(-\theta)}{\cos(-\theta)}$$

SOLUTION:

$$\begin{aligned}\frac{\tan(-\theta)}{\cos(-\theta)} &= \frac{-\tan\theta}{\cos\theta} \\ &= \frac{-\dfrac{\sin\theta}{\cos\theta}}{\cos\theta} \\ &= -\frac{\sin\theta}{\cos^2\theta}\end{aligned}$$

VERIFYING TRIGONOMETRIC IDENTITIES

The process of verifying trigonometric identities is similar to simplifying trigonometric expressions except that we know in advance the desired result. Remember to use the suggestions we offered at the beginning of this chapter when verifying trigonometric identities.

EXAMPLE: Verify the identity

$$\frac{\sin \theta}{\csc \theta} + \frac{\cos \theta}{\sec \theta} = 1$$

SOLUTION:

$$\frac{\sin \theta}{\csc \theta} + \frac{\cos \theta}{\sec \theta} = \frac{\sin \theta}{\frac{1}{\sin \theta}} + \frac{\cos \theta}{\frac{1}{\cos \theta}}$$

$$= (\sin \theta)(\sin \theta) + (\cos \theta)(\cos \theta)$$

$$= \sin^2\theta + \cos^2\theta$$

$$= 1$$

EXAMPLE: Verify the identity

$$1 + \cot^2 2x = \frac{1}{\sin^2 2x}$$

SOLUTION:

$$1 + \cot^2 2x = \csc^2 2x$$

$$= \frac{1}{\sin^2 2x}$$

EXAMPLE: Verify the identity

$$2 \sec \theta = \frac{\cos(-\theta)}{1 - \sin(-\theta)} + \frac{\cos(-\theta)}{1 + \sin(-\theta)}$$

SOLUTION: $\dfrac{\cos(-\theta)}{1 - \sin(-\theta)} + \dfrac{\cos(-\theta)}{1 + \sin(-\theta)}$

$$= \frac{\cos\theta}{1 + \sin\theta} + \frac{\cos\theta}{1 - \sin\theta}$$

$$= \frac{(\cos\theta)(1 - \sin\theta) + (\cos\theta)(1 + \sin\theta)}{(1 + \sin\theta)(1 - \sin\theta)}$$

$$= \frac{\cos\theta - \cos\theta\sin\theta + \cos\theta + \cos\theta\sin\theta}{1 + \sin\theta - \sin\theta - \sin^2\theta}$$

$$= \frac{2\cos\theta}{1 - \sin^2\theta}$$

$$= \frac{2\cos\theta}{\cos^2\theta}$$

$$= \frac{2}{\cos\theta}$$

$$= 2\sec\theta$$

EXAMPLE: Verify the identity

$$\frac{1 + 2\sin\theta + \sin^2\theta}{\cos^2\theta} = \frac{\sec\theta + \tan\theta}{\sec\theta - \tan\theta}$$

SOLUTION:

$$\frac{\sec\theta + \tan\theta}{\sec\theta - \tan\theta} = \frac{\dfrac{1}{\cos\theta} + \dfrac{\sin\theta}{\cos\theta}}{\dfrac{1}{\cos\theta} - \dfrac{\sin\theta}{\cos\theta}}$$

$$= \frac{\dfrac{1 + \sin\theta}{\cos\theta}}{\dfrac{1 - \sin\theta}{\cos\theta}}$$

$$= \left(\frac{1 + \sin\theta}{\cos\theta}\right)\left(\frac{\cos\theta}{1 - \sin\theta}\right)$$

$$= \frac{1 + \sin\theta}{1 - \sin\theta}$$

$$= \left(\frac{1 + \sin\theta}{1 - \sin\theta}\right)\left(\frac{1 + \sin\theta}{1 + \sin\theta}\right)$$

$$= \frac{1 + \sin\theta + \sin\theta + \sin^2\theta}{1 - \sin\theta + \sin\theta - \sin^2\theta}$$

$$= \frac{1 + 2\sin\theta + \sin^2\theta}{1 - \sin^2\theta}$$

$$= \frac{1 + 2\sin\theta + \sin^2\theta}{\cos^2\theta}$$

PRACTICE PROBLEMS:

Verify the following identities:

1. $\dfrac{1}{\tan^2 x + 1} = \cos^2 x$

2. $\csc\theta - \sin\theta = \cos\theta\cot\theta$

3. $\dfrac{\sin^2\theta}{1 + \cos\theta} = 1 - \cos\theta$

4. $\sin^2\theta = [\cos(-\theta)][\sec(-\theta) - \cos(-\theta)]$

5. $1 - \cos^2 x = (\tan^2 x)(1 - \sin^2 x)$

6. $\dfrac{1 - \cos^2\theta}{\csc\theta} = \sin^3\theta$

7. $\dfrac{1}{2 + \cot^2(-\theta)} = \dfrac{1}{2\csc^2(-\theta) - \cot^2(-\theta)}$

NOTE: No ANSWERS are furnished since the result is known in advance for each of the preceding PRACTICE PROBLEMS.

FORMULAS FOR IDENTITIES

In this section we will discuss the trigonometric formulas for the sum and difference of angles, for double angles, and for half angles.

SUM AND DIFFERENCE FORMULAS

The fundamental identities discussed in the previous section involved functions of a single angle. In this section we will consider identities involving functions of more than one angle.

We will start by developing a formula for cos $(\alpha - \beta)$. Refer to figure 6-1. Angles α and β are constructed in standard position, so angle *KOL* is equal to α and angle *KOM* is equal to β. We will also construct angle *KON* equal to $\alpha - \beta$. Since triangles *KON* and *MOL* are similar triangles, then sides *LM* and *KN* have the same length.

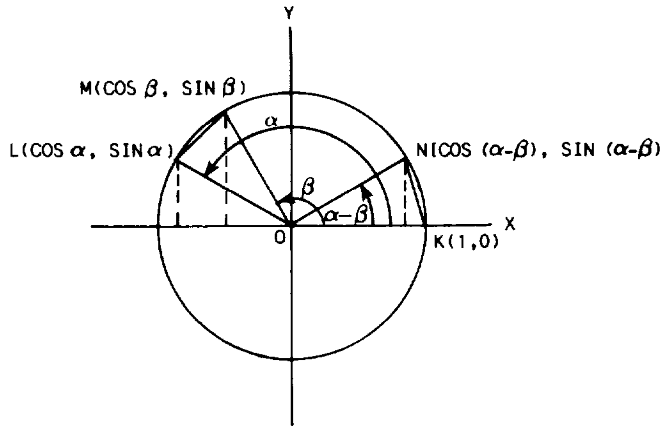


Figure 6-1.—Developing formula for cos $(\alpha - \beta)$.

Now we need to determine the coordinates of points *K, L, M,* and *N*. Recall the properties of right triangles, quadrantal angles, and reduction formulas. For the unit circle, where $r = 1$, the coordinates of point *K*, which lie on the positive *X* axis, are (1,0). According to properties of right triangles

$$\cos \theta = \frac{x}{r}$$

and

$$\sin \theta = \frac{y}{r}$$

So the coordinates of point *N* are $[\cos (\alpha - \beta), \sin (\alpha - \beta)]$.

Recall from chapter 4 that

$$\cos (180^\circ - \theta) = -\cos \theta$$

and

$$\sin (180^\circ - \theta) = \sin \theta$$

where θ is a positive acute angle. If we apply these formulas to angles α and β and note that the coordinates of a point in the second quadrant are $(-x,y)$, then the coordinates of point *L* are $(\cos \alpha, \sin \alpha)$ and the coordinates of point *M* are $(\cos \beta, \sin \beta)$.

Using the coordinates of these points and the distance formula, we can determine the lengths of sides LM and KN. Hence,

$$(LM)^2 = (\cos \alpha - \cos \beta)^2 + (\sin \alpha - \sin \beta)^2$$

$$= \cos^2\alpha - 2 \cos \alpha \cos \beta + \cos^2\beta$$

$$+ \sin^2\alpha - 2 \sin \alpha \sin \beta + \sin^2\beta$$

$$= 2 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta$$

and

$$(KN)^2 = [1 - \cos (\alpha - \beta)]^2 + [0 - \sin (\alpha - \beta)]^2$$

$$= 1 - 2 \cos (\alpha - \beta) + \cos^2(\alpha - \beta) + \sin^2(\alpha - \beta)$$

$$= 2 - 2 \cos (\alpha - \beta)$$

Since sides LM and KN have the same length, we can equate the distances and simplify as follows:

$$2 - 2 \cos \alpha \cos \beta - 2 \sin \alpha \sin \beta = 2 - 2 \cos (\alpha - \beta)$$

$$-2(\cos \alpha \cos \beta + \sin \alpha \sin \beta) = -2 \cos (\alpha - \beta)$$

$$\cos \alpha \cos \beta + \sin \alpha \sin \beta = \cos (\alpha - \beta)$$

Therefore, *the cosine of the difference of two angles is equal to the cosine of the first angle times the cosine of the second angle plus the sine of the first angle times the sine of the second angle*; that is,

$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

EXAMPLE: Simplify $\cos (90° - \beta)$.

SOLUTION: If

$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

then

$$\cos (90° - \beta) = \cos 90° \cos \beta + \sin 90° \sin \beta$$

$$= (0)(\cos \beta) + (1)(\sin \beta)$$

$$= \sin \beta$$

which is the same result shown in chapter 4.

EXAMPLE: Determine cos 15° using the cosine of the difference of two angles.

SOLUTION:

$$\cos 15^\circ = \cos (45^\circ - 30^\circ)$$

$$= \cos 45^\circ \cos 30^\circ + \sin 45^\circ \sin 30^\circ$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) + \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$

$$= \frac{\sqrt{6} + \sqrt{2}}{4}$$

To develop a formula for $\cos (\alpha + \beta)$, we will substitute $(-\beta)$ for β into the formula for $\cos (\alpha - \beta)$ as follows:

$$\cos (\alpha + \beta) = \cos [\alpha - (-\beta)]$$

$$= \cos \alpha \cos (-\beta) + \sin \alpha \sin (-\beta)$$

$$= \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

Therefore, *the cosine of the sum of two angles is equal to the cosine of the first angle times the cosine of the second angle minus the sine of the first angle times the sine of the second angle;* that is,

$$\cos (\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

EXAMPLE: Determine cos 105° using the cosine of the sum of two angles.

SOLUTION:

$$\cos 105^\circ = \cos (45^\circ + 60^\circ)$$

$$= \cos 45^\circ \cos 60^\circ - \sin 45^\circ \sin 60^\circ$$

$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{\sqrt{2} - \sqrt{6}}{4}$$

We will now use the identities

$$\cos \theta = \sin (90° - \theta)$$

and

$$\sin \theta = \cos (90° - \theta)$$

and the formula

$$\cos (\alpha - \beta) = \cos \alpha \cos \beta + \sin \alpha \sin \beta$$

to develop a formula for $\sin (\alpha + \beta)$ as follows:

$$\begin{aligned} \sin (\alpha + \beta) &= \cos [90° - (\alpha + \beta)] \\ &= \cos [(90° - \alpha) - \beta] \\ &= \cos (90° - \alpha) \cos \beta + \sin (90° - \alpha) \sin \beta \\ &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \end{aligned}$$

Therefore, *the sine of the sum of two angles is equal to the sine of the first angle times the cosine of the second angle plus the cosine of the first angle times the sine of the second angle;* that is,

$$\sin (\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

EXAMPLE: Verify that

$$\sin (\alpha + 45°) = \frac{\sqrt{2}}{2}(\sin \alpha + \cos \alpha)$$

SOLUTION:

$$\begin{aligned} \sin (\alpha + 45°) &= \sin \alpha \cos 45° + \cos \alpha \sin 45° \\ &= (\sin \alpha)\left(\frac{\sqrt{2}}{2}\right) + (\cos \alpha)\left(\frac{\sqrt{2}}{2}\right) \\ &= \frac{\sqrt{2}}{2}(\sin \alpha + \cos \alpha) \end{aligned}$$

Substituting $(-\beta)$ for β into the formula for $\sin (\alpha + \beta)$ produces

$$\begin{aligned} \sin (\alpha - \beta) &= \sin [\alpha + (-\beta)] \\ &= \sin \alpha \cos (-\beta) + \cos \alpha \sin (-\beta) \\ &= \sin \alpha \cos \beta - \cos \alpha \sin \beta \end{aligned}$$

Therefore, *the sine of the difference of two angles is equal to the sine of the first angle times the cosine of the second angle minus the cosine of the first angle times the sine of the second angle*; that is,

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

EXAMPLE: Use the formula for the sine of the difference of two angles to determine the value of

$$\sin 40^\circ \cos 10^\circ - \cos 40^\circ \sin 10^\circ$$

SOLUTION:

$$\begin{aligned}\sin 40^\circ \cos 10^\circ - \cos 40^\circ \sin 10^\circ &= \sin(40^\circ - 10^\circ)\\ &= \sin 30^\circ\\ &= 1/2\end{aligned}$$

Now, using the identity

$$\tan\theta = \frac{\sin\theta}{\cos\theta}$$

and the formulas for $\sin(\alpha + \beta)$ and $\cos(\alpha + \beta)$, we can develop a formula for $\tan(\alpha + \beta)$ as follows:

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\sin(\alpha + \beta)}{\cos(\alpha + \beta)}\\ &= \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}\end{aligned}$$

Dividing both the numerator and denominator by $\cos\alpha\cos\beta$ gives

$$\begin{aligned}\tan(\alpha + \beta) &= \frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta} + \dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta} - \dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}\\ &= \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}\end{aligned}$$

Therefore, *the tangent of the sum of two angles is equal to the quantity of the tangent of the first angle plus the tangent of the* *second angle divided by the quantity of 1 minus the tangent of the first angle times the tangent of the second angle*; that is,

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

EXAMPLE: If $\sin\alpha = -4/5$ and $\cos\beta = 12/13$, where α is in quadrant III and β is in quadrant IV, find $\tan(\alpha + \beta)$.

SOLUTION: Refer to figure 6-2. If $\sin\alpha = -4/5$ and α is in quadrant III, then $\tan\alpha = 4/3$. Likewise, if $\cos\beta = 12/13$ and β is in quadrant IV, then $\tan\beta = -5/12$. Therefore,

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}$$

$$= \frac{(4/3) + (-5/12)}{1 - (4/3)(-5/12)}$$

$$= \frac{11/12}{1 + 20/36}$$

$$= \left(\frac{11}{12}\right)\left(\frac{36}{56}\right)$$

$$= \frac{33}{56}$$

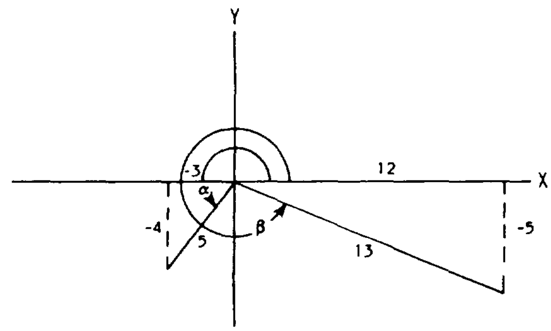


Figure 6-2.—Triangles in quadrants III and IV.

As before, to develop a formula for $\tan(\alpha - \beta)$, we will substitute $(-\beta)$ for β into the formula for $\tan(\alpha + \beta)$ as follows:

$$\tan(\alpha - \beta) = \tan[\alpha + (-\beta)]$$

$$= \frac{\tan\alpha + \tan(-\beta)}{1 - \tan\alpha\tan(-\beta)}$$

$$= \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

Therefore, *the tangent of the difference of two angles is equal to the quantity of the tangent of the first angle minus the tangent of the second angle divided by the quantity of 1 plus the tangent of the first angle times the tangent of the second angle*; that is,

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}$$

EXAMPLE: If $\csc \alpha = 29/21$, $\sin \beta = -8/17$, $\cos \alpha$ is negative, and $\sec \beta$ is positive, find $\cot (\alpha - \beta)$.

SOLUTION: Note that

$$\cot (\alpha - \beta) = \frac{1}{\tan (\alpha - \beta)}$$

So we determine the value of $\cot (\alpha - \beta)$ using the formula for $\tan (\alpha - \beta)$. Since $\csc \alpha$ is positive and $\cos \alpha$ is negative in quadrant II, then $\tan \alpha = -21/20$. Likewise, since $\sin \beta$ is negative and $\sec \beta$ is positive in quadrant IV, then $\tan \beta = -8/15$. Hence,

$$\begin{aligned}
\tan (\alpha - \beta) &= \frac{\tan \alpha - \tan \beta}{1 + \tan \alpha \tan \beta} \\
&= \frac{(-21/20) - (-8/15)}{1 + (-21/20)(-8/15)} \\
&= \frac{-31/60}{1 + 14/25} \\
&= \left(\frac{-31}{60}\right)\left(\frac{25}{39}\right) \\
&= -\frac{155}{468}
\end{aligned}$$

Therefore,

$$\cot (\alpha - \beta) = -\frac{468}{155}$$

PRACTICE PROBLEMS:

Use sum and difference formulas to find the values of the following:

1. $\sin \frac{13\pi}{12}$

2. $\cot 165°$

Verify the following using sum and difference formulas:

3. $\dfrac{\cos(\alpha - \beta)}{\cos\alpha \sin\beta} = \cot\beta + \tan\alpha$

4. $\tan\left(\alpha - \dfrac{\pi}{4}\right) = \dfrac{\tan\alpha - 1}{\tan\alpha + 1}$

5. If $\sin\alpha = -1/4$ and $\cos\beta = -4/5$, where α and β are both in quadrant III, find $\cos(\alpha + \beta)$.

ANSWERS:

1. $\dfrac{-\sqrt{6} + \sqrt{2}}{4}$

2. $\dfrac{1 + \sqrt{3}}{1 - \sqrt{3}}$ or $-2 - \sqrt{3}$

3. Result is known

4. Result is known

5. $\dfrac{4\sqrt{15} - 3}{20}$

DOUBLE-ANGLE FORMULAS

Formulas for the functions of twice an angle may be derived from the functions of the sum of two angles. Setting $\beta = \alpha$ in the formulas for $\sin(\alpha + \beta)$, $\cos(\alpha + \beta)$, and $\tan(\alpha + \beta)$ gives the following results:

$$\sin(\alpha + \alpha) = \sin\alpha \cos\alpha + \cos\alpha \sin\alpha$$

$$\cos(\alpha + \alpha) = \cos\alpha \cos\alpha - \sin\alpha \sin\alpha$$

$$\tan(\alpha + \alpha) = \frac{\tan\alpha + \tan\alpha}{1 - \tan\alpha \tan\alpha}$$

Hence,

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\begin{aligned}\cos 2\alpha &= \cos^2\alpha - \sin^2\alpha \\ &= 1 - 2 \sin^2\alpha \\ &= 2 \cos^2\alpha - 1\end{aligned}$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2\alpha}$$

The previous formulas are known as the *double-angle formulas*.

EXAMPLE: Find the values for $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, if $\tan \theta = -12/5$ and θ is in the second quadrant.

SOLUTION: Since θ is in the second quadrant, then $\sin \theta = 12/13$ and $\cos \theta = -5/13$; so,

$$\begin{aligned}\sin 2\theta &= 2 \sin \theta \cos \theta \\ &= 2\left(\frac{12}{13}\right)\left(\frac{-5}{13}\right) \\ &= -\frac{120}{169}\end{aligned}$$

$$\begin{aligned}\cos 2\theta &= \cos^2\theta - \sin^2\theta \\ &= \left(\frac{-5}{13}\right)^2 - \left(\frac{12}{13}\right)^2 \\ &= \frac{25}{169} - \frac{144}{169} \\ &= -\frac{119}{169}\end{aligned}$$

$$\begin{aligned}\tan 2\theta &= \frac{2 \tan \theta}{1 - \tan^2\theta} \\ &= \frac{2(-12/5)}{1 - (-12/5)^2} \\ &= \frac{-24/5}{1 - 144/25} \\ &= \frac{-24/5}{-119/25} \\ &= \left(\frac{24}{5}\right)\left(\frac{25}{119}\right) \\ &= \frac{120}{119}\end{aligned}$$

EXAMPLE: Verify that

$$\csc 2x = \frac{1}{2} \csc x \sec x$$

SOLUTION:

$$\frac{1}{2} \csc x \sec x = \frac{1}{2}\left(\frac{1}{\sin x}\right)\left(\frac{1}{\cos x}\right)$$

$$= \frac{1}{2 \sin x \cos x}$$

$$= \frac{1}{\sin 2x}$$

$$= \csc 2x$$

HALF-ANGLE FORMULAS

From the double-angle formulas we can derive the *half-angle formulas*. Since

$$\cos 2\alpha = 1 - 2 \sin^2\alpha$$

then solving for $\sin \alpha$ results in

$$\sin \alpha = \pm \sqrt{\frac{1 - \cos 2\alpha}{2}}$$

Now, if $2\alpha = \theta$, so that $\alpha = \theta/2$, then

$$\sin \frac{\theta}{2} = \pm \sqrt{\frac{1 - \cos \theta}{2}}$$

which is the half-angle formula for $\sin \theta/2$.

The half-angle formula for $\cos \theta/2$ can be obtained by solving

$$\cos 2\alpha = 2 \cos^2\alpha - 1$$

for $\cos \alpha$ or

$$\cos \alpha = \pm \sqrt{\frac{1 + \cos 2\alpha}{2}}$$

As before, if $2\alpha = \theta$, so that $\alpha = \theta/2$, then

$$\cos \frac{\theta}{2} = \pm \sqrt{\frac{1 + \cos \theta}{2}}$$

The half-angle formula for tan $\theta/2$ is derived from the half-angle formulas for sine and cosine as follows:

$$\tan\frac{\theta}{2} = \frac{\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}}$$

$$= \frac{\pm\sqrt{\dfrac{1-\cos\theta}{2}}}{\pm\sqrt{\dfrac{1+\cos\theta}{2}}}$$

$$= \pm\sqrt{\frac{1-\cos\theta}{1+\cos\theta}}$$

NOTE: For the half-angle formulas, the positive or negative sign is selected according to the quadrant in which $\theta/2$ lies.

EXAMPLE: Use the half-angle formulas to find the cosine, sine, and tangent of 112.5°.

SOLUTION: Since 112.5° lies in quadrant II, the cosine and tangent will be negative and the sine will be positive; so,

$$\cos 112.5° = -\sqrt{\frac{1+\cos 225°}{2}}$$

$$= -\sqrt{\frac{1+(-\sqrt{2}/2)}{2}}$$

$$= -\sqrt{\frac{2-\sqrt{2}}{4}}$$

$$= -\frac{\sqrt{2-\sqrt{2}}}{2}$$

$$\sin 112.5° = \sqrt{\frac{1-\cos 225°}{2}}$$

$$= \sqrt{\frac{1-(-\sqrt{2}/2)}{2}}$$

$$= \sqrt{\frac{2+\sqrt{2}}{4}}$$

$$= \frac{\sqrt{2+\sqrt{2}}}{2}$$

$$\tan 112.5° = -\sqrt{\frac{1 - \cos 225°}{1 + \cos 225°}}$$

$$= -\sqrt{\frac{1 - (-\sqrt{2}/2)}{1 + (-\sqrt{2}/2)}}$$

$$= -\sqrt{\frac{1 + \sqrt{2}/2}{1 - \sqrt{2}/2}}$$

$$= -\sqrt{\frac{2 + \sqrt{2}}{2 - \sqrt{2}}}$$

$$= -\sqrt{\left(\frac{2 + \sqrt{2}}{2 - \sqrt{2}}\right)\left(\frac{2 + \sqrt{2}}{2 + \sqrt{2}}\right)}$$

$$= -\sqrt{\frac{6 + 4\sqrt{2}}{2}}$$

$$= -\sqrt{3 + 2\sqrt{2}}$$

EXAMPLE: Verify that

$$\sin^2\left(\frac{\theta}{2}\right) = \frac{\sec\theta - 1}{2\sec\theta}$$

SOLUTION:

$$\frac{\sec\theta - 1}{2\sec\theta} = \frac{\frac{1}{\cos\theta} - 1}{2\left(\frac{1}{\cos\theta}\right)}$$

$$= \frac{\frac{1 - \cos\theta}{\cos\theta}}{\frac{2}{\cos\theta}}$$

$$= \frac{1 - \cos\theta}{2}$$

$$= \sin^2\left(\frac{\theta}{2}\right)$$

PRACTICE PROBLEMS:

1. Find the values for $\sin 2\theta$, $\cos 2\theta$, and $\tan 2\theta$, if $\theta = \pi$.

2. Find the values for $\sin \theta/2$, $\cos \theta/2$, and $\tan \theta/2$ if $\sec \theta = 17/8$, $\tan \theta$ is positive, and $0 \leq \theta \leq 360°$.

Verify the following using double-angle and half-angle formulas:

3. $(1 + \tan x)(\tan 2x) = \dfrac{2 \tan x}{1 - \tan x}$

4. $\dfrac{2}{1 + \cos \theta} - \tan^2\left(\dfrac{\theta}{2}\right) = 1$

ANSWERS:

1. $\sin 2\theta = 0$

   $\cos 2\theta = 1$

   $\tan 2\theta = 0$

2. $\sin \theta/2 = 3\sqrt{34}/34$

   $\cos \theta/2 = 5\sqrt{34}/34$

   $\tan \theta/2 = 3/5$

3. Result is known

4. Result is known

INVERSE TRIGONOMETRIC FUNCTIONS

In this section we will discuss the notations that apply to the inverse trigonometric functions along with the principal values of the inverse functions.

NOTATION

Let us consider the inverse of the sine function, $y = \sin x$. The inverse of the sine function may be denoted as

$$x = \sin y$$

or

$$y = \sin^{-1}x$$

which can be read "the inverse sine of x." Note that $\sin^{-1}x$ does not mean $1/\sin x$.

The inverse of the sine function may also be denoted by

$$y = \arcsin x$$

which can be read "the arc sine of x." The notation $\arcsin x$ arises because it is the length of an arc on the unit circle for which the sine is x.

Similar notation occurs for the inverses of the other trigonometric functions; that is, $\cos^{-1}x$ or $\arccos x$, $\tan^{-1}x$ or $\arctan x$, etc.

PRINCIPAL VALUES

For any angle, one, and only one, value of a trigonometric function corresponds to the angle; but for any value of a trigonometric function, numerous angles satisfy the value. Hence, the inverses of the trigonometric functions are not themselves functions. However, if we restrict the ranges of these relationships, we can obtain functions. The values of the trigonometric functions in the restricted ranges are called *principal values*. To indicate this restriction, we will capitalize the first letter in the name of the inverse trigonometric function; that is,

$$y = \text{Sin}^{-1}x$$

or

$$y = \text{Arcsin } x$$

Table 6-1.—Inverse Trigonometric Functions

FUNCTION	DOMAIN	RANGE
$y = \text{Sin}^{-1}x$	$-1 \leq x \leq 1$	$-\pi/2 \leq y \leq \pi/2$
$y = \text{Cos}^{-1}x$	$-1 \leq x \leq 1$	$0 \leq y \leq \pi$
$y = \text{Tan}^{-1}x$	Any real number	$-\pi/2 < y < \pi/2$
$y = \text{Cot}^{-1}x$	Any real number	$0 < y < \pi$
$y = \text{Sec}^{-1}x$	$x \leq -1$ or $x \geq 1$	$0 \leq y \leq \pi$, $y \neq \pi/2$
$y = \text{Csc}^{-1}x$	$x \leq -1$ or $x \geq 1$	$-\pi/2 \leq y \leq \pi/2$, $y \neq 0$

and so on for all the trigonometric functions. Table 6-1 shows the six inverse trigonometric functions, their domains, and their ranges.

EXAMPLE: Find all values of arctan 1.

SOLUTION: The tangent of many angles is 1, such as $\pi/4$, $5\pi/4$, $9\pi/4$, and $13\pi/4$. Thus the values of arctan 1 are

$$\frac{\pi}{4} + n\pi$$

where n is any integer.

EXAMPLE: Find Arctan 1.

SOLUTION: In the restricted range, as shown in table 6-1, the only number whose tangent is 1 is $\pi/4$. Hence,

$$\text{Arctan } 1 = \pi/4$$

EXAMPLE: Find Arcsec 2.236 in degrees.

SOLUTION: As previously determined,

$$\sec \theta = \frac{1}{\cos \theta}$$

If

$$x = \sec \theta$$

then

$$x = \frac{1}{\cos \theta}$$

or

$$\cos\theta = \frac{1}{x}$$

If we solve for θ in the above equations, then

$$\theta = \text{arcsec } x$$

and

$$\theta = \arccos\frac{1}{x}$$

so,

$$\text{arcsec } x = \arccos\frac{1}{x}$$

Hence, for the given problem

$$\text{Arcsec } 2.236 = \text{Arccos}\left(\frac{1}{2.236}\right)$$

$$= \text{Arccos } 0.44723$$

According to appendix II, the angle whose cosine is 0.44723 is $63°26'$ to the nearest minute, which is in the range $0 \leq y \leq 180°$. Therefore,

$$\text{Arcsec } 2.236 = 63°\,26'$$

EXAMPLE: Find $\text{Cos}^{-1}(-0.50000)$ in degrees.

SOLUTION: According to appendix II, the angle whose cosine is 0.50000 is 60°. However, we want the angle whose cosine is a negative number so that the angle is in the range of $0 \leq y \leq 180°$. Since the cosine of a number is negative in the second quadrant where the reference angle of 60° corresponds to 120°, then

$$\text{Cos}^{-1}(-0.50000) = 120°$$

EXAMPLE: Find $\mathrm{Cot}^{-1}(-\sqrt{3})$ in degrees.

SOLUTION: The expression $\mathrm{Cot}^{-1}(-\sqrt{3})$ can be interpreted as "the angle between 0 and π, whose cotangent is $-\sqrt{3}$." Recall that

$$\cot\theta = \frac{\text{adjacent}}{\text{opposite}}$$

For the given problem,

$$\theta = \mathrm{Cot}^{-1}(-\sqrt{3})$$

or

$$\mathrm{Cot}\,\theta = -\sqrt{3}$$

From our previous discussion of special angles, you should recognize the reference angle of θ to be 30°. Since the cotangent of an angle is negative in the second quadrant for the range from 0° to 180°, then θ is 150°; that is,

$$\mathrm{Cot}^{-1}(-\sqrt{3}) = 150^\circ$$

EXAMPLE: Evaluate $\cos\left(\mathrm{Arcsin}\,\frac{5}{13}\right)$.

SOLUTION: Let

$$u = \mathrm{Arcsin}\,\frac{5}{13}$$

so

$$\mathrm{Sin}\,u = \frac{5}{13}$$

Since Arcsin is defined only in quadrants I and IV and since 5/13 is positive, then u is in quadrant I. Figure 6-3 shows a triangle in quadrant I whose sine of angle u is 5/13. Using the Pythagorean theorem, we find that the side adjacent to angle u is 12. Therefore,

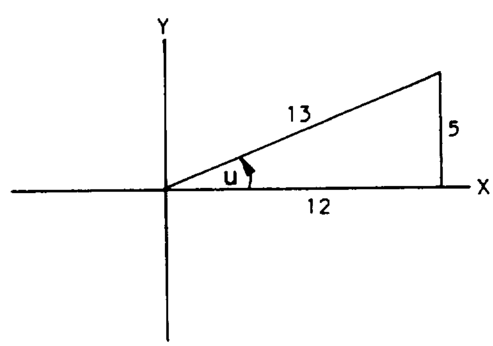


Figure 6-3.—Triangle in quadrant I.

$$\cos u = \frac{12}{13}$$

or

$$\cos\left(\text{Arcsin } \frac{5}{13}\right) = \frac{12}{13}$$

EXAMPLE: Evaluate $\sin\left(\text{Arccos } \frac{12}{13} - \text{Arcsin } \frac{4}{5}\right)$.

SOLUTION: Let

$$u = \text{Arccos } \frac{12}{13}$$

or

$$\text{Cos } u = \frac{12}{13}$$

and let

$$v = \text{Arcsin } \frac{4}{5}$$

or

$$\text{Sin } v = \frac{4}{5}$$

Angles u and v would both be in quadrant I according to previous conditions. Figure 6-4, view A, shows a triangle in quadrant I where cos u = 12/13. Figure 6-4, view B, shows a triangle in quadrant I where sin v = 4/5.

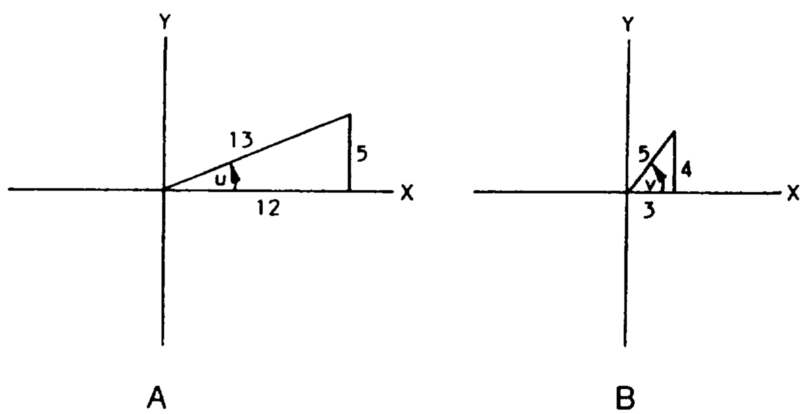


Figure 6-4.—Triangles in quadrant I.

Our given equation is in the form of the difference formula

$$\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$$

where

$$\sin\left(\text{Arccos}\,\frac{12}{13} - \text{Arcsin}\,\frac{4}{5}\right)$$

$$= \sin\left(\text{Arccos}\,\frac{12}{13}\right)\cos\left(\text{Arcsin}\,\frac{4}{5}\right)$$

$$- \cos\left(\text{Arccos}\,\frac{12}{13}\right)\sin\left(\text{Arcsin}\,\frac{4}{5}\right)$$

Referring to figure 6-4, view A, we find that

$$\sin\left(\text{Arccos}\,\frac{12}{13}\right) = \frac{5}{13}$$

and

$$\cos\left(\text{Arccos}\,\frac{12}{13}\right) = \frac{12}{13}$$

Referring to figure 6-4, view B, we find that

$$\sin\left(\text{Arcsin}\,\frac{4}{5}\right) = \frac{4}{5}$$

and

$$\cos\left(\text{Arcsin}\,\frac{4}{5}\right) = \frac{3}{5}$$

Hence,

$$\sin\left(\text{Arccos}\,\frac{12}{13} - \text{Arcsin}\,\frac{4}{5}\right) = \left(\frac{5}{13}\right)\left(\frac{3}{5}\right) - \left(\frac{12}{13}\right)\left(\frac{4}{5}\right)$$

$$= \frac{15 - 48}{65}$$

$$= -\frac{33}{65}$$

PRACTICE PROBLEMS:

1. Find $\text{Cos}^{-1}(1/2)$.
2. Find Arcsin 0.88295 in degrees.
3. Find $\text{Csc}^{-1}(-1.57208)$ in degrees.
4. Find Arccot $(-\sqrt{3}/3)$.
5. Evaluate tan [Arcsin $(-1/2)$].
6. Evaluate cot [$\text{Cos}^{-1}(-0.19994)$].
7. Evaluate cos (Arctan 5/12 − Arccot 4/3).
8. Evaluate sin [$\text{Sec}^{-1}(-25/24) - \text{Csc}^{-1}(-17/8)$].

ANSWERS:

1. $\pi/3$
2. $62°$
3. $-39° 30'$
4. $2\pi/3$
5. $-1/\sqrt{3}$ or $-\sqrt{3}/3$
6. -0.20406
7. $63/65$
8. $-87/425$

TRIGONOMETRIC EQUATIONS

A *trigonometric equation* is an equality that is true for some values but may not be true for all values of the variable. The principles and processes used to solve algebraic equations may be used to solve trigonometric equations. The identities and formulas previously studied may also be used in solving trigonometric equations.

The following suggestions may be helpful to you in solving trigonometric equations:

1. *If only one trigonometric function is present, solve the equation for that function.*
2. *If more than one function is present, rearrange the equation so that one side equals 0. Then try to factor and set each factor equal to zero to solve. You may find it helpful to use identities and formulas to change the form of the equation or to square both sides of the equation.*
3. *If the equation is quadratic in form, but not factorable, use the quadratic formula.*
4. *All possible solutions should be tested in the given equation.*

EXAMPLE: Solve $\tan\theta - 1 = 0$ for $0° \leq \theta < 360°$.

SOLUTION: We can rewrite

$$\tan\theta - 1 = 0$$

to read

$$\tan\theta = 1$$

or

$$\theta = \arctan 1$$

and solve the equation. Therefore, the solutions in the given interval are

$$\theta = 45° \text{ and } 225°$$

EXAMPLE: Solve $\sin 2\theta = 2\cos\theta$ for $0° \leq \theta < 360°$.

SOLUTION: We will rearrange the equation so that one side equals 0; hence,

$$\sin 2\theta - 2\cos\theta = 0$$

Since,

$$\sin 2\theta = 2\sin\theta\cos\theta$$

we will substitute this formula to change the form of our equation; that is,

$$2 \sin \theta \cos \theta - 2 \cos \theta = 0$$

We will now factor $2 \cos \theta$ from each term, so

$$2 \cos \theta (\sin \theta - 1) = 0$$

and set each factor equal to zero where

$$2 \cos \theta = 0$$

and

$$\sin \theta - 1 = 0$$

Solving each term gives

$$2 \cos \theta = 0$$

or

$$\theta = \arccos 0$$

where

$$\theta = 90° \text{ and } 270°$$

and

$$\sin \theta - 1 = 0$$

$$\sin \theta = 1$$

or

$$\theta = \arcsin 1$$

where

$$\theta = 90°$$

Substituting the value of 90° into the given equation gives

$$\sin 2(90^\circ) = 2 \cos 90^\circ$$

$$\sin 180^\circ = (2)(0)$$

$$0 = 0$$

Substituting the value of 270° into the given equation gives

$$\sin 2(270^\circ) = 2 \cos 270^\circ$$

$$\sin 540^\circ = 2(0)$$

$$0 = 0$$

Therefore, $\theta = 90^\circ$ and 270° are the solutions to the equation.

EXAMPLE: Solve $\tan x - \sec x + 1 = 0$ for $0 \leq x < 2\pi$.

SOLUTION: Rewrite the given equation as

$$\tan x + 1 = \sec x$$

Square both sides of the equation to get

$$(\tan x + 1)^2 = (\sec x)^2$$

$$\tan^2 x + 2 \tan x + 1 = \sec^2 x$$

$$(\tan^2 x + 1) + 2 \tan x = \sec^2 x$$

Note that $\tan^2 x + 1 = \sec^2 x$, so

$$2 \tan x = 0$$

$$\tan x = 0$$

or

$$x = \arctan 0$$

Hence, the possible solutions are 0 and π. Substituting 0 into the original equation gives

$$\tan 0 + 1 = \sec 0$$

$$0 + 1 = 1$$

$$1 = 1$$

Substituting π into the original equation gives

$$\tan \pi + 1 = \sec \pi$$

$$0 + 1 = -1$$

but

$$1 \neq -1$$

Therefore, the only solution to the given equation is 0.

EXAMPLE: Solve $\cot^2\theta - 3 \cot \theta - 2 = 0$ for $0° \leq \theta < 360°$.

SOLUTION: Since this equation cannot be factored, we will use the quadratic formula, introduced in *Mathematics*, Volume 1,

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where the values for x are possible solutions to the equation

$$ax^2 + bx + c = 0$$

For our equation

$$x = \cot \theta$$

$$a = 1$$

$$b = -3$$

$$c = -2$$

Hence,

$$\cot \theta = \frac{-(-3) \pm \sqrt{(-3)^2 - 4(1)(-2)}}{2(1)}$$

$$= \frac{3 \pm \sqrt{17}}{2}$$

So,

$$\cot \theta = 3.56155$$

or

$$\theta = \text{arccot } 3.56155$$

where

$$\theta = 15°\,41' \text{ and } 195°\,41'$$

to the nearest minute in quadrants I and III, respectively. And

$$\cot\theta = -0.56155$$

or

$$\theta = \text{arccot}\,(-0.56155)$$

where

$$\theta = 119°\,19' \text{ and } 299°\,19'$$

to the nearest minute in quadrants II and IV, respectively. Substituting all four of the values of

$$\theta = 15°\,41',\ 119°\,19',\ 195°\,41',\ 299°\,19'$$

into the original equation shows that they are solutions.

NOTE: When substituting a possible solution into the original equation, we may not be able to equate the sides exactly because of rounding error.

PRACTICE PROBLEMS:

1. Solve $\sin\theta = -\sqrt{3}/2$ for $0° \le \theta < 360°$.
2. Solve $\tan x \cos^2 x = \sin^2 x$ for $0 \le x < 2\pi$.
3. Solve $\cot\theta - \csc\theta - \sqrt{3} = 0$ for $0° \le \theta < 360°$.
4. Solve $7\sin^2\theta - 3\sin\theta - 4 = 0$ for $0° \le \theta < 360°$.

ANSWERS:

1. $\theta = 240°$ and $300°$
2. $x = 0, \pi/4, \pi, 5\pi/4$
3. $\theta = 240°$
4. $\theta = 90°$, $214°\,51'$, and $325°\,9'$

SUMMARY

The following are the major topics covered in this chapter:

1. **Suggestions in solving identities:**

   1. Know the basic identities.
   2. Attempt to transform the more complicated side into the other side.
   3. When possible, express all trigonometric functions in the equation in terms of sine and cosine.
   4. Perform any factoring or algebraic operations.

2. **Reciprocal identities:**

$$\cot \theta = \frac{1}{\tan \theta}$$

$$\sec \theta = \frac{1}{\cos \theta}$$

$$\csc \theta = \frac{1}{\sin \theta}$$

3. **Quotient identities:**

$$\tan \theta = \frac{\sin \theta}{\cos \theta}$$

$$\cot \theta = \frac{\cos \theta}{\sin \theta}$$

4. **Pythagorean identities:**

$$\cos^2\theta + \sin^2\theta = 1$$

$$1 + \tan^2\theta = \sec^2\theta$$

$$1 + \cot^2\theta = \csc^2\theta$$

5. **Identities for negative angles:**

$$\sin(-\theta) = -\sin \theta$$

$$\cos(-\theta) = \cos \theta$$

$$\tan(-\theta) = -\tan \theta$$

6. **Sum and difference formulas:**

The cosine of the difference of two angles is equal to the cosine of the first angle times the cosine of the second angle plus the sine of the first angle times the sine of the second angle; that is,

$$\cos(\alpha - \beta) = \cos\alpha \cos\beta + \sin\alpha \sin\beta$$

The cosine of the sum of two angles is equal to the cosine of the first angle times the cosine of the second angle minus the sine of the first angle times the sine of the second angle; that is,

$$\cos(\alpha + \beta) = \cos\alpha \cos\beta - \sin\alpha \sin\beta$$

The sine of the sum of two angles is equal to the sine of the first angle times the cosine of the second angle plus the cosine of the first angle times the sine of the second angle; that is,

$$\sin(\alpha + \beta) = \sin\alpha \cos\beta + \cos\alpha \sin\beta$$

The sine of the difference of two angles is equal to the sine of the first angle times the cosine of the second angle minus the cosine of the first angle times the sine of the second angle; that is,

$$\sin(\alpha - \beta) = \sin\alpha \cos\beta - \cos\alpha \sin\beta$$

The tangent of the sum of two angles is equal to the quantity of the tangent of the first angle plus the tangent of the second angle divided by the quantity of 1 minus the tangent of the first angle times the tangent of the second angle; that is,

$$\tan(\alpha + \beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha \tan\beta}$$

The tangent of the difference of two angles is equal to the quantity of the tangent of the first angle minus the tangent of the second angle divided by the quantity of 1 plus the tangent of the first angle times the tangent of the second angle; that is

$$\tan(\alpha - \beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha \tan\beta}$$

7. **Double-angle formulas:**

$$\sin 2\alpha = 2 \sin \alpha \cos \alpha$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha$$

$$= 1 - 2 \sin^2\alpha$$

$$= 2 \cos^2\alpha - 1$$

$$\tan 2\alpha = \frac{2 \tan \alpha}{1 - \tan^2\alpha}$$

8. **Half-angle formulas:**

$$\sin \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{2}}$$

$$\cos \frac{\theta}{2} = \pm\sqrt{\frac{1 + \cos \theta}{2}}$$

$$\tan \frac{\theta}{2} = \pm\sqrt{\frac{1 - \cos \theta}{1 + \cos \theta}}$$

9. **Inverse trigonometric functions:**

$x = \sin y$ or $y = \sin^{-1}x$ or $y = \arcsin x$

$x = \cos y$ or $y = \cos^{-1}x$ or $y = \arccos x$

$x = \tan y$ or $y = \tan^{-1}x$ or $y = \arctan x$

$x = \cot y$ or $y = \cot^{-1}x$ or $y = \text{arccot}\ x$

$x = \sec y$ or $y = \sec^{-1}x$ or $y = \text{arcsec}\ x$

$x = \csc y$ or $y = \csc^{-1}x$ or $y = \text{arccsc}\ x$

10. **Principal values:** The values of the trigonometric functions in the restricted ranges are called *principal values.* This restriction is indicated by the capitalization of the first letter of the name of the inverse trigonometric function.

11. **Trigonometric equations:** A *trigonometric equation* is an equality that is true for some values but may not be true for all values of the variable.

12. **Suggestions in solving trigonometric equations:**

1. If only one trigonometric function is present, solve the equation for that function.
2. If more than one function is present, rearrange the equation so that one side equals 0. Then try to factor and set each factor equal to zero to solve. You may find it helpful to use identities and formulas to change the form of the equation or to square both sides of the equation.
3. If the equation is quadratic in form, but not factorable, use the quadratic formula.
4. All possible solutions should be tested in the given equation.

ADDITIONAL PRACTICE PROBLEMS

1. Verify that $\frac{1}{1 - \cos x} + \frac{1}{1 + \cos x} = 2 \csc^2 x$.

2. Verify that $\frac{\sin (x + y)}{\cos (x - y)} = \frac{\cot x + \cot y}{1 + \cot x \cot y}$ using sum and difference formulas.

3. If $\tan \alpha = 8/15$ with α in quadrant I and $\cos \beta = 7/25$ with β in quadrant IV, find $\sec (\alpha - \beta)$.

4. Verify that $\tan x = \frac{1 - \cos 2x}{\sin 2x}$ using double-angle formulas.

5. Verify that $8 \sin^2\left(\frac{x}{2}\right) \cos^2\left(\frac{x}{2}\right) = 1 - \cos 2x$ using half-angle formulas.

6. Find Arcsec (-2).

7. Find $\text{Tan}^{-1}(-0.12278)$ in degrees.

8. Evaluate $\sin [2 \text{ Tan}^{-1}(12/5)]$. HINT: $\sin 2\theta = 2 \sin \theta \cos \theta$.

9. Evaluate $\tan \left[\text{Arccos } \frac{\sqrt{3}}{2} - \text{Arcsin } \left(\frac{-3}{5}\right)\right]$.

10. Solve $\sin x + \cos x = \sqrt{2}$ if $0 \leq x < 2\pi$.

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. Result is known
2. Result is known
3. $-425/87$
4. Result is known
5. Result is known
6. $2\pi/3$
7. $-7°$
8. $120/169$
9. $\dfrac{4 + 3\sqrt{3}}{4\sqrt{3} - 3}$ or $\dfrac{25\sqrt{3} + 48}{39}$
10. $\pi/4$

CHAPTER 7

VECTORS AND FORCES

LEARNING OBJECTIVES

Upon completion of this chapter, you should be able to do the following:

1. Add, subtract, and determine the components of vectors.

2. Solve problems involving forces.

3. Solve problems involving translational and rotational equilibrium.

INTRODUCTION

The last chapter in this course deals with vectors and forces. Any study of vectors and forces requires a knowledge of trigonometry.

VECTORS

A *scalar quantity* is one that has magnitude only; that is, 10 watts, 4 miles, 17 acres, and 28.2 pounds per square inch. A *vector quantity* is one that has both magnitude and direction; that is, 6 miles due north, 250 knots at 30°, and 400 miles per hour to the west. Scalar quantities are represented by italicized letters. Vector quantities are represented by placing arrows over the italicized letters, for instance, $\vec{A}$; and the magnitude of the vector quantity is represented by the italicized letter of the vector quantity. Vectors are geometrically represented by arrows. The arrowhead represents the terminal end of a vector and indicates the vector's direction. The other end of the vector is called the initial end. The magnitude of the vector is the vector's length.

Two vectors are said to be equal if they are of the same length, are parallel, and point in the same direction. In figure 7-1, view A, $\vec{A} = \vec{B}$.

If two vectors have the same length, are parallel, but point in opposite directions, they are said to be negatives of each other. In figure 7-1, view B, $\vec{C} = -\vec{D}$ and $\vec{D} = -\vec{C}$.

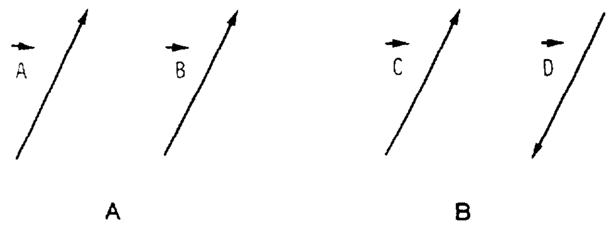


Figure 7-1.—Equal and negative vectors.

Hence, a vector can be moved from one position to another without being changed if its direction and magnitude are kept unchanged.

VECTOR ADDITION

The general rule for adding vectors is illustrated in figure 7-2. To add $\vec{B}$ to $\vec{A}$, shift $\vec{B}$ until its initial end coincides with the terminal end of $\vec{A}$. In its new position, $\vec{B}$ will be parallel, the same length, and in the same direction it was in the old position of $\vec{B}$. To find the vector sum of $\vec{A} + \vec{B}$, draw a vector, $\vec{R}$, with its initial end at the initial end of $\vec{A}$ and its terminal end at the terminal end of $\vec{B}$. Hence, $\vec{R}$ is called the *resultant vector* of $\vec{A}$ and $\vec{B}$, which is written

$$\vec{R} = \vec{A} + \vec{B}$$

If we reverse the process to add $\vec{A}$ to $\vec{B}$, we would move $\vec{A}$ until its initial end coincides with the terminal end of $\vec{B}$ so that

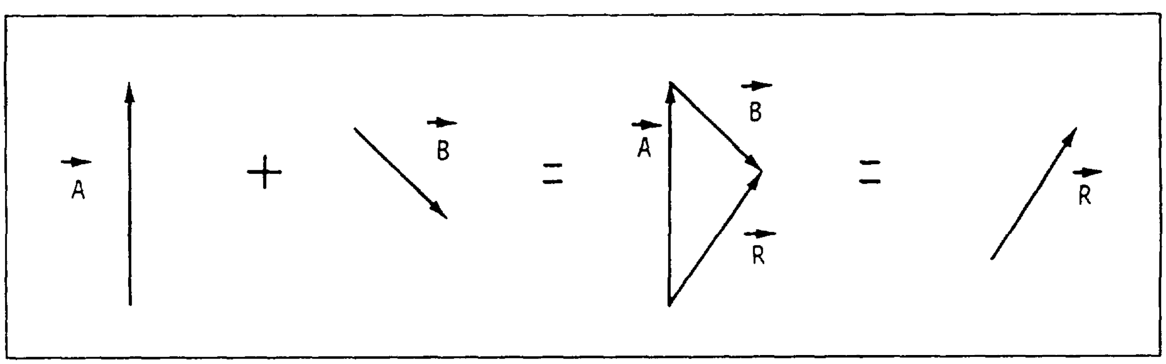


Figure 7-2.—Addition of $\vec{B}$ to $\vec{A}$.

$\vec{R}$ is also the resultant vector of $\vec{B} + \vec{A}$. (See fig. 7-3.) Thus

$$\vec{R} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$$

To find the resultant vector of any number of vectors, place the initial end of each vector to the terminal end of the previous vector. Be sure the new vector position is parallel, the same length, and in the same direction as the old position. Draw a vector $\vec{R}$, from the initial end of the first vector to the terminal end of the last vector so that $\vec{R}$ is the resultant vector. Figure 7-4 shows how four vectors are added together. We recognize again that the order in which the vectors are added does not affect the result. (See fig. 7-5.) Thus,

$$\vec{R} = \vec{A} + \vec{B} + \vec{C} + \vec{D} = \vec{D} + \vec{C} + \vec{B} + \vec{A}$$

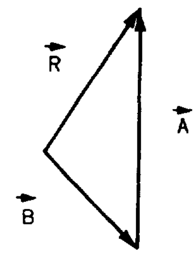


Figure 7-3.—Addition of $\vec{A}$ to $\vec{B}$.

VECTOR SUBTRACTION

We can also subtract one vector from another vector. Refer to figure 7-6. To subtract $\vec{B}$ from $\vec{A}$, we need to determine

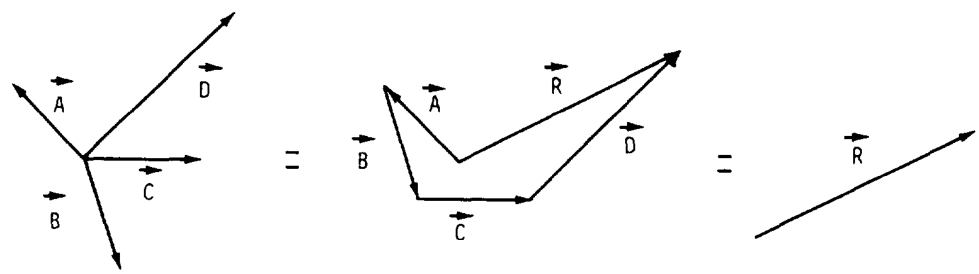


Figure 7-4.—Addition of $\vec{A} + \vec{B} + \vec{C} + \vec{D}$.

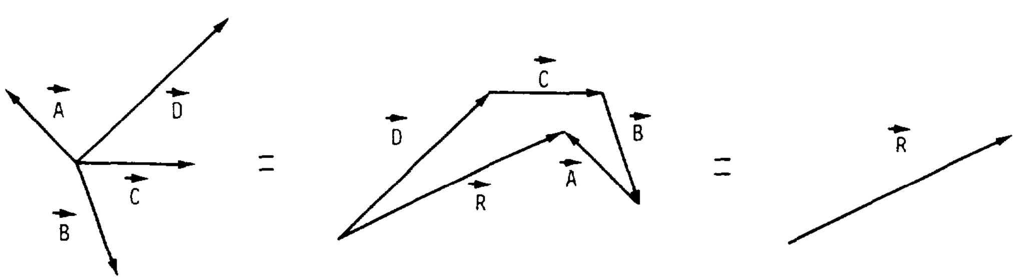


Figure 7-5.—Addition of $\vec{D} + \vec{C} + \vec{B} + \vec{A}$.

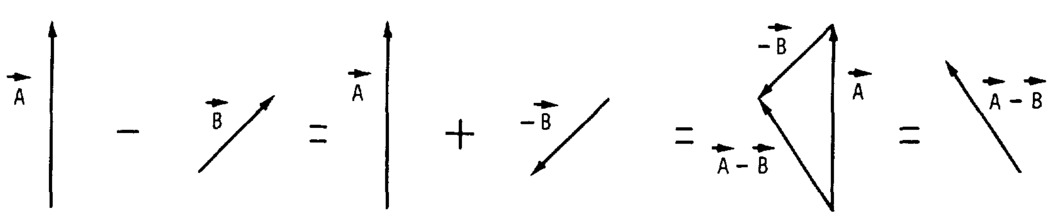


Figure 7-6.—Vector subtraction.

the negative of $\vec{B}$, denoted by $-\vec{B}$. The negative of a vector is a vector that is parallel, the same length, and points in the opposite direction. Hence, we add $-\vec{B}$ to $\vec{A}$ as previously discussed. This process may be summarized as

$$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

COMPONENTS OF VECTORS

The projections of a vector onto the X and Y axes of the rectangular coordinate system are called the *components* of a vector. We say that a vector is resolved into its x and y components, called the *horizontal* and *vertical components* of a vector, respectively. In figure 7-7, view A,

$$\vec{V}_x = \text{horizontal component of } \vec{V}$$

and

$$\vec{V}_y = \text{vertical component of } \vec{V}$$

Figure 7-7, view B, shows the magnitudes of $\vec{V}$, $\vec{V}_x$, and $\vec{V}_y$. Using properties of right triangles, we see that

$$\cos\theta = \frac{V_x}{V}$$

or

$$V_x = V\cos\theta$$

and

$$\sin\theta = \frac{V_y}{V}$$

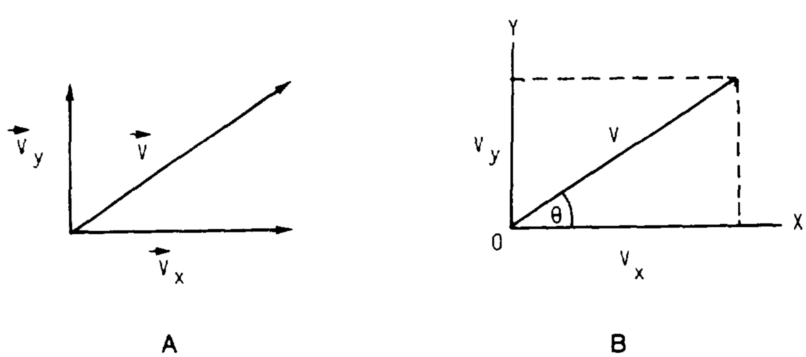


Figure 7-7.—Components of a vector.

or

$$V_y = V \sin \theta$$

where V_x, V_y, and V are the magnitudes of $\vec{V}_x$, $\vec{V}_y$, and $\vec{V}$, respectively. Notice also that (by use of the Pythagorean theorem) *the magnitude of V can be found to be*

$$V^2 = V_x^2 + V_y^2$$

or

$$V = \sqrt{V_x^2 + V_y^2}$$

The direction of $\vec{V}$ is the angle, θ, the vector makes with the horizontal. This direction can be determined by

$$\tan \theta = \frac{V_y}{V_x}$$

or

$$\theta = \arctan \frac{V_y}{V_x}$$

The direction of $\vec{V}_x$ is 0° and the direction of $\vec{V}_y$ is 90°.

EXAMPLE: Find the magnitude of the horizontal and vertical components of a vector having a magnitude of 50 pounds acting at an angle of 30° to the horizontal.

SOLUTION:

$$\begin{aligned} V_x &= V \cos \theta \\ &= 50 \cos 30° \\ &= 50\left(\frac{\sqrt{3}}{2}\right) \\ &= 25\sqrt{3} \\ &= 43.3 \text{ pounds (rounded)} \end{aligned}$$

and

$$\begin{aligned} V_y &= V \sin \theta \\ &= 50 \sin 30° \\ &= 50\left(\frac{1}{2}\right) \\ &= 25 \text{ pounds} \end{aligned}$$

EXAMPLE: Find the magnitude and direction of a vector whose horizontal and vertical components have a magnitude of 90 newtons and 60 newtons, respectively.

SOLUTION: The magnitude of the resultant vector is

$$\begin{aligned} V &= \sqrt{V_x^2 + V_y^2} \\ &= \sqrt{(90)^2 + (60)^2} \\ &= \sqrt{8{,}100 + 3{,}600} \\ &= \sqrt{11{,}700} \\ &= 108.2 \text{ newtons (rounded)} \end{aligned}$$

The direction of the resultant vector is

$$\begin{aligned} \theta &= \arctan \frac{60}{90} \\ &= \arctan 0.66667 \\ &= 33^\circ\, 41' \text{ (to the nearest minute)} \end{aligned}$$

VECTOR ADDITION BY COMPONENTS

We can add vectors that lie in the same plane by working in terms of their components. This procedure is as follows:

1. *Resolve the given vectors into their x and y components.*
2. *Add the magnitudes of the x components to give R_x (the magnitude of the x component of $\vec{R}$), and add the magnitudes of the y components to give R_y (the magnitude the y component of $\vec{R}$); that is,*

   $$R_x = A_x + B_x + C_x + \dots$$

   and

   $$R_y = A_y + B_y + C_y + \dots$$

3. *Find the magnitude and direction of $\vec{R}$ from R_x and R_y. The magnitude can be determined by the use of the Pythagorean theorem; that is,*

   $$R = \sqrt{R_x^2 + R_y^2}$$

   The direction of $\vec{R}$ can be found from the values of the components by trigonometry; that is,

   $$\theta = \arctan \frac{R_y}{R_x}$$

EXAMPLE: A girl walks 110 feet south, 100 feet east, 120 feet northeast, and then 90 feet northwest. What is the magnitude and direction from her starting point?

SOLUTION: For convenience we will call north the positive y direction, south the negative y direction, east the positive x direction, and west the negative x direction on a rectangular coordinate system. Refer to figure 7-8, view A, for the path the girl walks. Figure 7-8, view B, shows the magnitude and direction of each vector from the origin according to the rectangular coordinate system. Hence, the magnitudes of the components of $\vec{A}$ are

$$\begin{aligned} A_x &= A \cos \theta \\ &= 110 \cos 270^\circ \\ &= 110(0) \\ &= 0 \text{ feet} \end{aligned}$$

and

$$\begin{aligned} A_y &= A \sin \theta \\ &= 110 \sin 270^\circ \\ &= 110(-1) \\ &= -110 \text{ feet} \end{aligned}$$

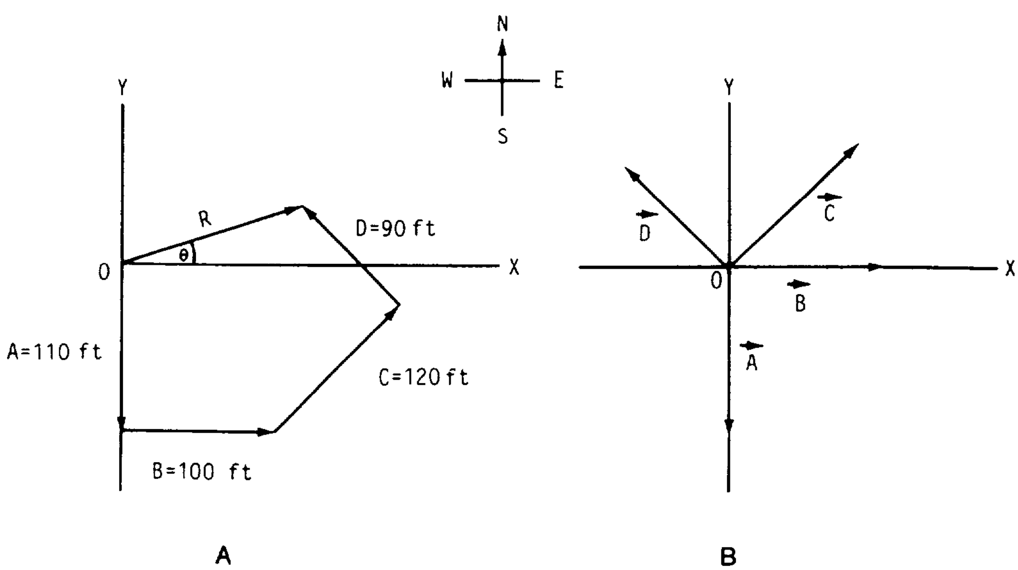


Figure 7-8.—Vector addition by components.

The magnitudes of the components of $\vec{B}$ are

$$\begin{aligned} B_x &= B \cos \theta \\ &= 100 \cos 0^\circ \\ &= 100(1) \\ &= 100 \text{ feet} \end{aligned}$$

and

$$\begin{aligned} B_y &= B \sin \theta \\ &= 100 \sin 0^\circ \\ &= 100(0) \\ &= 0 \text{ feet} \end{aligned}$$

The magnitudes of the components of $\vec{C}$ are

$$\begin{aligned} C_x &= C \cos \theta \\ &= 120 \cos 45^\circ \\ &= 120\left(\frac{\sqrt{2}}{2}\right) \\ &= 60\sqrt{2} \text{ feet} \end{aligned}$$

and

$$\begin{aligned} C_y &= C \sin \theta \\ &= 120 \sin 45^\circ \\ &= 120\left(\frac{\sqrt{2}}{2}\right) \\ &= 60\sqrt{2} \text{ feet} \end{aligned}$$

And the magnitudes of the components of $\vec{D}$ are

$$\begin{aligned} D_x &= D \cos \theta \\ &= 90 \cos 135^\circ \\ &= -90 \cos 45^\circ \\ &= -90\left(\frac{\sqrt{2}}{2}\right) \\ &= -45\sqrt{2} \text{ feet} \end{aligned}$$

and

$$
\begin{aligned}
D_y &= D \sin \theta \\
&= 90 \sin 135^\circ \\
&= 90 \sin 45^\circ \\
&= 90\left(\frac{\sqrt{2}}{2}\right) \\
&= 45\sqrt{2} \text{ feet}
\end{aligned}
$$

Now we will add the magnitudes of the x components to get R_x and add the magnitudes of the y components to get R_y:

$$
\begin{aligned}
R_x &= A_x + B_x + C_x + D_x \\
&= 0 + 100 + 60\sqrt{2} - 45\sqrt{2} \\
&= 121.21 \text{ feet (rounded)}
\end{aligned}
$$

and

$$
\begin{aligned}
R_y &= A_y + B_y + C_y + D_y \\
&= -110 + 0 + 60\sqrt{2} + 45\sqrt{2} \\
&= 38.49 \text{ feet (rounded)}
\end{aligned}
$$

Therefore, the magnitude of the resultant vector from the girl's starting point is

$$
\begin{aligned}
R &= \sqrt{R_x^2 + R_y^2} \\
&= \sqrt{(121.21)^2 + (38.49)^2} \\
&= \sqrt{16{,}173.34} \\
&= 127.17 \text{ feet (rounded)}
\end{aligned}
$$

and the direction from her starting point is

$$
\begin{aligned}
\theta &= \arctan \frac{R_y}{R_x} \\
&= \arctan \frac{38.49}{121.21} \\
&= \arctan 0.31755 \\
&= 17^\circ\, 37' \text{ north of east}
\end{aligned}
$$

PRACTICE PROBLEMS:

Give magnitude accuracy to one decimal place and angle accuracy to the nearest minute for the following problems:

1. Find V_x and V_y of $\vec{V}$ having a magnitude of 325 pounds making an angle of 78° 20′ with the horizontal component vector.

2. Two vectors, having magnitudes of 150 newtons and 220 newtons, act at right angles to each other. Find the magnitude of their resultant vector and the angle it makes with the larger vector.

3. An airplane is heading due east with an airspeed (speed relative to the air) of 350 mph. A wind is blowing from due south at 58 mph.

   a. Find the airplane's angle of drift (the angle between its heading and its actual course).

   b. Find the ground speed (actual speed along its course).

4. Given three vectors with magnitudes and directions of 40 feet, 60°; 60 feet, 150°; and 80 feet, 225°, find the magnitude and the direction of the resultant vector.

ANSWERS:

1. V_x = 65.7 pounds

   V_y = 318.3 pounds

2. V = 266.3 newtons

   θ = 34° 17′

3. a. 9° 25′ north of east

   b. 354.8 mph

4. V = 88.9 feet

   θ = 174° 50′

FORCES

A *force* produces or prevents motion or has the tendency to do so. The effect of a force upon a body depends upon the magnitude and direction of the force. Therefore, *a force can be represented by a vector quantity*. The *resolution of a force*, then, is the separation of a single force into two or more component forces acting in given directions on the same point. Moreover, when two or more forces act on the same body, the *resultant force* is the single force whose effect upon the body is equal in magnitude and direction to the combined effects of all the forces acting on the body.

EXAMPLE: Two dogs on leashes held by a person are trying to move in directions perpendicular to each other, one pulling with a force of 64 pounds, the other with a 52-pound force. Find the magnitude of the resultant force and the angle it makes with the larger force vector.

SOLUTION: Refer to figure 7-9. If we let $\vec{F}$ be the force vector, then $\vec{F_x}$ and $\vec{F_y}$ are the two components at right angles to each other. If point A is the person holding the two dogs, $\overrightarrow{AB}$ the larger force vector, and $\overrightarrow{AD}$ the smaller force vector, then

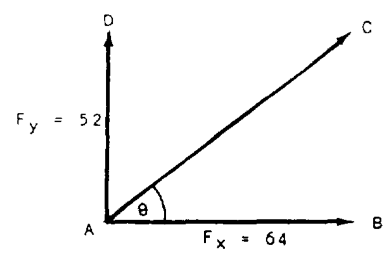


Figure 7-9.—Force vectors.

$$\begin{aligned} F &= \sqrt{F_x^2 + F_y^2} \\ &= \sqrt{(AB)^2 + (AD)^2} \\ &= \sqrt{(64)^2 + (52)^2} \\ &= \sqrt{6{,}800} \\ &= 82.5 \text{ pounds (rounded)} \end{aligned}$$

and

$$\begin{aligned} \theta &= \arctan \frac{F_y}{F_x} \\ &= \arctan \frac{52}{64} \\ &= \arctan 0.81250 \\ &= 39^\circ\ 6' \end{aligned}$$

EXAMPLE: An automobile weighing 3,200 pounds is parked on a driveway that makes a 17° angle with the horizontal. Find the components of the car's weight parallel and perpendicular to the driveway.

SOLUTION: The weight of an object is the gravitational force the earth exerts on it, which is a force that acts vertically downward. (See fig. 7-10.) Since $\vec{F} = \vec{W}$ is vertical and $\vec{F}_x$ is perpendicular to the driveway, then the angle between $\vec{F}$ and $\vec{F}_x$ is also $\theta = 17°$. Hence,

$$
\begin{aligned}
F_x &= F \cos \theta \\
&= W \cos \theta \\
&= 3{,}200 \cos 17° \\
&= 3{,}200(0.95630) \\
&= 3{,}060.2 \text{ pounds (rounded)}
\end{aligned}
$$

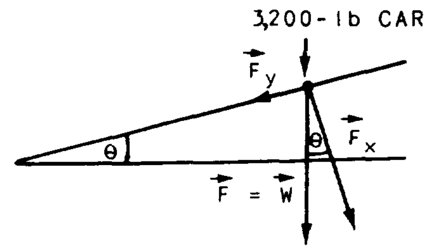


Figure 7-10.—Gravitational force.

which is the car's weight perpendicular to the driveway, and

$$
\begin{aligned}
F_y &= F \sin \theta \\
&= W \sin \theta \\
&= 3{,}200 \sin 17° \\
&= 3{,}200(0.29237) \\
&= 935.6 \text{ pounds (rounded)}
\end{aligned}
$$

which is the car's weight parallel to the driveway.

PRACTICE PROBLEMS:

1. A force of 235 pounds makes an angle of 60° 40′ with the vertical. Resolve the force into its horizontal and vertical components. Give component accuracy to one decimal place.

2. Two forces, 15 pounds at 335° and 25 pounds at 14°, act on the same point. Determine the magnitude (to one decimal place) and direction (to the nearest minute) of the resultant force.

3. A force of 53.5 pounds is acting on an object at a 25° angle to the horizontal. Find the horizontal and vertical components of the force. Give component accuracy to two decimal places.

ANSWERS:

1. $\vec{F}_x = 204.9$ pounds

   $\vec{F}_y = 115.1$ pounds

2. $F = 37.9$ pounds

   $\theta = 359° \, 34'$

3. $\vec{F}_x = 48.49$ pounds

   $\vec{F}_y = 22.61$ pounds

EQUILIBRIUM

If a body undergoes no change in its motion, it is said to be in a state of *equilibrium*. Two conditions are required for a body at rest to be in equilibrium. The body must have neither translatory (straight line) motion nor rotary (spinning) motion.

When two or more forces act together at a point, the *equilibriant force* is that single force applied at the same point which produces equilibrium. *The equilibriant force has a magnitude equal to that of the resultant of the separate forces, but it acts in the opposite direction.*

EXAMPLE: A force of 17 newtons at 123° and a force of 33 newtons at 333° act on the same point. Determine the magnitudes and directions of both the resultant and the equilibriant.

SOLUTION: For the force of 17 newtons at 123°, the magnitudes of the horizontal and vertical components are

$$A_x = 17 \cos 123°$$

$$= -17 \cos 57°$$

$$= -9.3 \text{ newtons (rounded)}$$

and

$$A_y = 17 \sin 123°$$

$$= 17 \sin 57°$$

$$= 14.3 \text{ newtons (rounded)}$$

For the force of 33 newtons at 333°, the magnitudes of the horizontal and vertical components are

$$B_x = 33 \cos 333°$$

$$= 33 \cos 27°$$

$$= 29.4 \text{ newtons (rounded)}$$

and

$$B_y = 33 \sin 333°$$

$$= -33 \sin 27°$$

$$= -15.0 \text{ newtons (rounded)}$$

Hence, the magnitudes of the horizontal and vertical components of the resultant vector are

$$F_x = A_x + B_x$$

$$= -9.3 + 29.4$$

$$= 20.1 \text{ newtons}$$

and

$$F_y = A_y + B_y$$

$$= 14.3 + -15.0$$

$$= -0.7 \text{ newtons}$$

Since the resultant and the equilibriant have equal magnitudes, then

$$F = \sqrt{F_x^2 + F_y^2}$$

$$= \sqrt{(20.1)^2 + (-0.7)^2}$$

$$= 20.1 \text{ newtons (rounded)}$$

The direction of the resultant is

$$\theta = \arctan \frac{F_y}{F_x}$$

$$= \arctan \frac{-0.7}{20.1}$$

$$= -2°$$

$$= 358°$$

(since our resultant force is in the fourth quadrant) and the direction of the equilibriant is

$$\theta' = \theta \pm 180°$$

$$= 358° - 180°$$

$$= 178°$$

TRANSLATIONAL EQUILIBRIUM

The first condition for equilibrium, no translatory motion, is met when no unbalanced forces act on a body. Therefore, *the sum of the forces acting on a body in any direction must be equal to the sum of the forces acting on a body in the opposite direction.*

Since the sum of all forces acting on a body must equal zero, then the sum of all the magnitudes of the horizontal components must equal zero and the sum of all the magnitudes of the vertical components must equal zero; that is,

$$F_x = 0$$

and

$$F_y = 0$$

EXAMPLE: Find the tension (a force acting against the resistance of a body) of a weightless rope supporting a 50-pound block.

SOLUTION: Refer to figure 7-11. Since there are no horizontal components in this problem, we only need to find the magnitude of the vertical component of the tension, $\vec{T}_y$, and the magnitude of the vertical component of the weight of the block, $\vec{W}_y$, such that

$$F_y = T_y + W_y = 0$$

or

$$T_y = -W_y$$

Since

$$W_y = W \sin \theta$$
$$= 50 \sin 270°$$
$$= -50 \text{ pounds}$$

then

$$T_y = -W_y$$
$$= -(-50)$$
$$= 50 \text{ pounds}$$

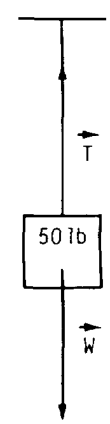
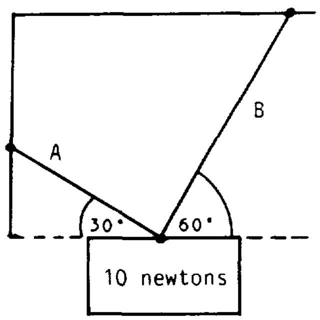


Figure 7-11.—Weight supported by a rope.

Therefore, the tension in the rope is equal to the weight being supported.

EXAMPLE: A weight of 10 newtons is supported by two cords. One cord makes an angle of 30° with the horizontal while the other makes an angle of 60° with the horizontal. Find the tension in each cord.



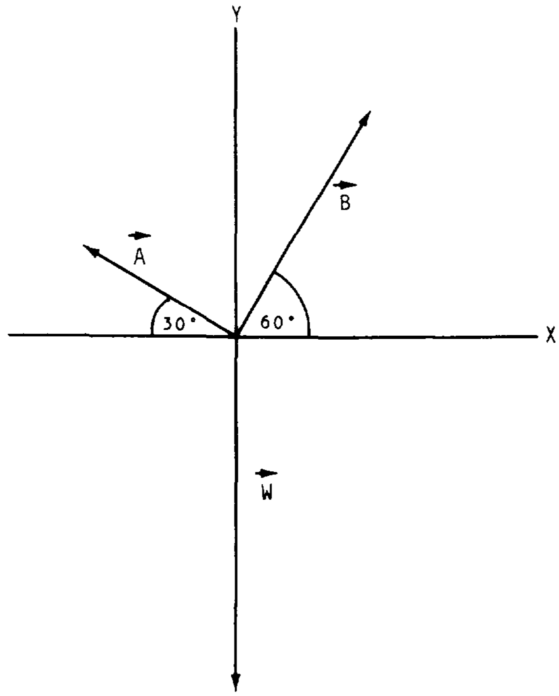


Figure 7-12.—Weight supported by two cords.

SOLUTION: Refer to figure 7-12. We begin by determining the magnitudes of the horizontal and vertical components of each vector. For tension $\vec{A}$, the reference angle (in the second quadrant) of sin θ is positive and the reference angle of cos θ is negative; so

$$A_x = -A \cos \theta$$

$$= -A \cos 30°$$

$$= -\frac{\sqrt{3}}{2}A \text{ newtons}$$

and

$$A_y = A \sin \theta$$

$$= A \sin 30°$$

$$= \frac{1}{2}A \text{ newtons}$$

For tension $\vec{B}$, the reference angles of $\sin \theta$ and $\cos \theta$ are both positive in quadrant I; so

$$\begin{aligned} B_x &= B \cos \theta \\ &= B \cos 60° \\ &= \frac{1}{2} B \text{ newtons} \end{aligned}$$

and

$$\begin{aligned} B_y &= B \sin \theta \\ &= B \sin 60° \\ &= \frac{\sqrt{3}}{2} B \text{ newtons} \end{aligned}$$

For the weight of 10 newtons,

$$\begin{aligned} W_x &= W \cos \theta \\ &= 10 \cos 270° \\ &= 0 \text{ newtons} \end{aligned}$$

and

$$\begin{aligned} W_y &= W \sin \theta \\ &= 10 \sin 270° \\ &= -10 \text{ newtons} \end{aligned}$$

Now, since the sum of the magnitudes of the horizontal components must equal zero and the sum of the magnitudes of the vertical components must equal zero, then

$$A_x + B_x + W_x = 0$$

or

$$-\frac{\sqrt{3}}{2} A + \frac{1}{2} B = 0$$

$$-\frac{\sqrt{3}}{2} A = -\frac{1}{2} B$$

$$\sqrt{3} A = B$$

and

$$A_y + B_y + W_y = 0$$

or

$$\frac{1}{2}A + \frac{\sqrt{3}}{2}B + -10 = 0$$

$$\frac{1}{2}A + \frac{\sqrt{3}}{2}B = 10$$

Substituting $\sqrt{3}A$ for B in the last equation, we obtain

$$\frac{1}{2}A + \frac{\sqrt{3}}{2}(\sqrt{3}A) = 10$$

$$\frac{1}{2}A + \frac{3}{2}A = 10$$

$$2A = 10$$

$$A = 5 \text{ newtons}$$

and

$$B = \sqrt{3}A$$

$$= \sqrt{3}(5)$$

$$= 8.7 \text{ newtons (rounded)}$$

ROTATIONAL EQUILIBRIUM

The second condition for equilibrium, no rotary motion, is met when the sum of the torques acting upon a body about a point equals zero; that is,

$$\tau_R = \tau_1 + \tau_2 + \tau_3 + \ldots = 0$$

Hence, *the sum of all the clockwise torques equals the sum of all the counterclockwise torques about an axis of rotation. Torque* is the product of the magnitude of a force, F, and the length of its torque or lever arm, L, where L is measured perpendicular to the line of action of the force. Hence,

$$\tau_R = F_1L_1 + F_2L_2 + F_3L_3 + \ldots = 0$$

If a force tends to produce a counterclockwise rotation about an axis, the torque will be considered positive. If a force tends to produce a clockwise rotation about an axis, the torque will be considered negative.

EXAMPLE: A person exerts a 15-pound force at the end of an 8-inch wrench. (See fig. 7-13.) If this force makes an angle of 45° with the handle, what is the torque produced on the nut?

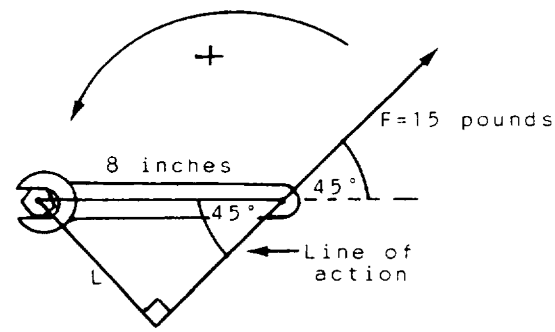


Figure 7-13.—Torque.

SOLUTION: First we need to find the length of the torque arm. Since L is measured perpendicular to the line of action, then to solve for L, we will use

$$\sin \theta = \frac{\text{opposite}}{\text{hypotenuse}}$$

or

$$\sin 45° = \frac{L}{8}$$

$$L = 8 \sin 45°$$

$$= 8\left(\frac{\sqrt{2}}{2}\right)$$

$$= 4\sqrt{2} \text{ inches}$$

Since the force tends to produce a counterclockwise rotation about the axis, then the torque produced on the nut is positive and

$$\tau = FL$$

$$= 15(4\sqrt{2})$$

$$= 60\sqrt{2}$$

$$= 84.9 \text{ pound} \cdot \text{inches (rounded)}$$

EXAMPLE: A rod 12 meters long has weights of 5 newtons and 15 newtons at its ends. (Assume that the weight of the rod is negligible.) At what point should the rod be picked up if it is to have no tendency to rotate (where is the balance point of the rod)?

SOLUTION: Refer to figure 7-14. First, compute the torques about the unknown balance point. If x is the distance of the 15-newton weight from this point, then the 5-newton weight is $(12 - x)$ meters from the point on the other side. Note that x is the length of the torque arm of the 15-newton weight and $(12 - x)$ is the length of the torque arm of the 5-newton weight. Since the 15-newton weight tends to produce a counterclockwise rotation about the balance point of the rod, then the torque will be considered positive. Likewise, since the 5-newton weight tends to produce a clockwise rotation about the balance point, then the torque will be considered negative. Hence,

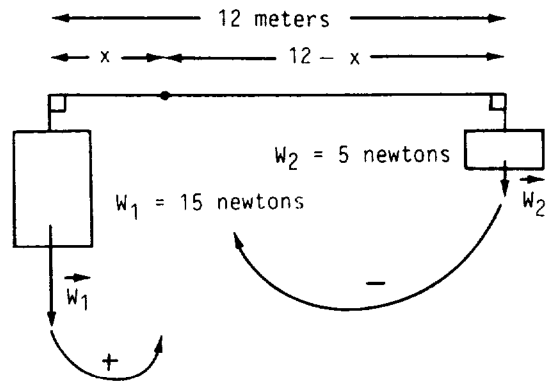


Figure 7-14.—Balanced rod.

$$\tau_1 = F_1L_1$$
$$= 15(x)$$
$$= 15x$$

and

$$\tau_2 = F_2L_2$$
$$= -5(12 - x)$$
$$= -60 + 5x$$

Since no rotary motion occurs when

$$\tau_R = \tau_1 + \tau_2 = 0$$

then

$$F_1L_1 + F_2L_2 = 0$$
$$15x - 60 + 5x = 0$$
$$20x = 60$$
$$x = 3 \text{ meters}$$

Therefore, when the rod is picked up 3 meters from the 15-newton weight end, the two weights exert opposite torques of the same magnitude $[15x = 45 = 5(12 - x)]$ about this point, where the rod is balanced.

PRACTICE PROBLEMS:

Give magnitude accuracy to one decimal place and angle accuracy to the nearest minute for the following:

1. Find the magnitudes and directions of the resultant and equilibriant forces of a force of 21 newtons due north and a second force of 32 newtons southeast.

2. A 20-pound ball is suspended by a rope, *A*, attached to a wall. Rope *A* is pulled away from the wall by a horizontal rope, *B*, and is held so that rope *A* forms an angle of 30° with the vertical wall. Find the tensions in ropes *A* and *B*.

3. A 150-newton force is applied to a pole 6 meters above its base at an angle of 45° above the horizontal. Find the torque about the base of the pole.

4. A uniform horizontal bar is 550 millimeters long and is of negligible weight. A 32-newton weight is hung from the left end of the bar, and a 70-newton weight is hung from the right end. Where should a single upward support be positioned to balance the system?

ANSWERS:

1. R = 22.7 newtons

   E = 22.7 newtons

   $\theta_R = 355° 57'$

   $\theta_E = 175° 57'$

2. A = 23.1 pounds

   B = 11.5 pounds

3. 636.4 newton · meters

4. 172.5 millimeters from the right end

SUMMARY

The following are the major topics covered in this chapter:

1. **Definitions:**

   A *scalar quantity* is one that has magnitude only.

   A *vector quantity* is one that has both magnitude and direction.

2. **Vectors:**

   Two vectors are said to be equal if they are of the same length, are parallel, and point in the same direction.

   If two vectors have the same length, are parallel, but point in opposite directions, they are said to be negatives of each other.

3. **Resultant vectors:** To find the resultant vector of any number of vectors, place the initial end of each vector to the terminal end of the previous vector. Be sure the new vector position is parallel, the same length, and in the same direction as the old vector position. Draw a vector from the initial end of the first vector to the terminal end of the last vector. This newly formed vector is the *resultant vector*.

4. **Vector addition:**

   $$\vec{R} = \vec{A} + \vec{B} = \vec{B} + \vec{A}$$

   where $\vec{A}$ and $\vec{B}$ are added and $\vec{R}$ is their resultant vector.

5. **Vector subtraction:**

   $$\vec{A} - \vec{B} = \vec{A} + (-\vec{B})$$

   where $\vec{B}$ is subtracted from $\vec{A}$, which is the same as adding the negative of $\vec{B}$ to $\vec{A}$.

6. **Components of vectors:** The projections of a vector onto the X and Y axes of the rectangular coordinate system are called the *components* of a vector. The *horizontal component* of $\vec{V}$ is $\vec{V}_x$ and the *vertical component* of $\vec{V}$ is $\vec{V}_y$.

7. **Magnitudes of vectors:** The magnitudes of $\vec{V}_x$, $\vec{V}_y$, and $\vec{V}$, respectively, are

$$V_x = V \cos \theta$$

$$V_y = V \sin \theta$$

$$V = \sqrt{V_x^2 + V_y^2}$$

8. **Directions of vectors:** The direction of $\vec{V}$ is the angle, θ, the vector makes with the horizontal.

$$\theta = \arctan \frac{V_y}{V_x}$$

The direction of $\vec{V}_x$ is 0° and the direction of $\vec{V}_y$ is 90°.

9. **Vector addition by components:**

   1. Resolve the given vectors into their x and y components.

   2. Add the magnitudes of the x components to give R_x (the magnitude of the x component of $\vec{R}$), and add the magnitudes of the y components to give R_y (the magnitude of the y component of $\vec{R}$); that is,

   $$R_x = A_x + B_x + C_x + \ldots$$

   and

   $$R_y = A_y + B_y + C_y + \ldots$$

   3. Find the magnitude and direction of $\vec{R}$ from R_x and R_y. The magnitude can be determined by the use of the Pythagorean theorem; that is,

   $$R = \sqrt{R_x^2 + R_y^2}$$

   The direction of $\vec{R}$ can be found from the values of the components by trigonometry; that is,

   $$\theta = \arctan \frac{R_y}{R_x}$$

10. **Force:** A *force* produces or prevents motion or has the tendency to do so.

    A force can be represented by a vector quantity.

The *resolution of a force* is the separation of a single force into two or more component forces acting in given directions on the same point.

When two or more forces act on the same body, the *resultant force* is the single force whose effect upon the body is equal in magnitude and direction to the combined effects of all the forces acting on the body.

11. **Equilibrium:** If a body undergoes no change in its motion, it is said to be in a state of *equilibrium.*

    For a body at rest to be in equilibrium, it must have neither translatory motion nor rotary motion.

12. **Equilibriant force:** When two or more forces act together at a point, the *equilibriant force* is that single force applied at the same point which produces equilibrium.

    The equilibriant force has a magnitude equal to that of the resultant of the separate forces, but it acts in the opposite direction.

13. **Translational equilibrium:** The sum of the forces acting on a body in any direction must be equal to the sum of the forces acting on a body in the opposite direction, such that

$$F_x = 0$$

and

$$F_y = 0$$

14. **Rotational equilibrium:** The sum of all the clockwise torques equals the sum of all the counterclockwise torques about an axis of rotation, such that,

$$\tau_R = \tau_1 + \tau_2 + \tau_3 + \ldots = F_1L_1 + F_2L_2 + F_3L_3 + \ldots = 0$$

    Torque is the product of the magnitude of force, *F*, and the length of its torque or lever arm, *L*, where *L* is measured perpendicular to the line of action of the force.

    If a force tends to produce a counterclockwise rotation about an axis, the torque will be considered positive. If a force tends to produce a clockwise rotation about an axis, the torque will be considered negative.

ADDITIONAL PRACTICE PROBLEMS

Give magnitude accuracy to one decimal place and angle accuracy to the nearest minute for the following:

1. An airplane is heading due south at 220 mph and the wind is blowing from due east at 65 mph. Find the airplane's angle of drift and its ground speed.

2. An airplane flies 100 miles west from city A to city B, then 100 miles north from city B to city C, and finally 50 miles southeast to city D. How far is it from city A to city D?

3. Three forces act simultaneously on a point. One force is 15 newtons at $0°$; the second is 20 newtons at $210°$; and the third is 30 newtons at $60°$. Determine the magnitude and direction of the resultant force.

4. A force of 11 pounds at $111°$ acts on a point. A second force of 22 pounds at $222°$ and a third force of 33 pounds at $333°$ also act on the same point. Determine the magnitude and direction of the equilibriant force.

5. A 100-pound tightrope walker stands at the center of a rope that is 200 feet in length. If the rope sags 20 feet at the center, find the tension in each side of the rope.

6. A 40-pound child and a 60-pound child sit at opposite ends of a 12-foot seesaw pivoted at its center. Where should a third child who weighs 50 pounds sit in order to balance the seesaw?

ANSWERS TO ADDITIONAL PRACTICE PROBLEMS

1. $\theta = 16^\circ\ 28'$ west of south

   $S = 229.4$ mph

2. 91.4 miles

3. $F = 20.4$ newtons

   $\theta = 51^\circ\ 34'$

4. $E = 21.5$ pounds

   $\theta = 115^\circ\ 22'$

5. 255 pounds

6. 2.4 feet from the pivot on the same side as the 40-pound child

Assignment 2

Textbook assignment: Chapter 3, "Trigonometric Measurements," pages 3-1 through 3-36 .

Learning Objective:

Apply angular measurement to problem solving.

In items 2-1 through 2-3, select from column B the expression most closely related to each term in column A.

	A. TERMS	B. EXPRESSIONS
2-1.	Radius vector	1. The original position of the radius vector
2-2.	Terminal position	2. The angle generated by rotating the radius vector counterclockwise from the initial position
2-3.	Positive angle	3. The line that is rotated to generate an angle
		4. The final position of the radius vector

2-4. One revolution is equal to 360

1. degrees
2. minutes
3. radians
4. seconds

2-5. If the angle generated by rotating the radius vector in a positive direction is in the third quadrant, the angle is between what two angles?

1. 0° and 90°
2. 90° and 180°
3. 180° and 270°
4. 270° and 360°

2-6. In which of the following quadrants does an angle of 710° lie?

1. First
2. Second
3. Third
4. Fourth

2-7. An angle of $1,320^\circ$ would be in which of the following quadrants?

1. First
2. Second
3. Third
4. Fourth

● In answering items 2-8 and 2-9, use the general formula for the relationship between the radian measure of an angle, the length of the arc it subtends, and the length of the radius vector; that is, $\theta = s/r$.

2-8. An angle that intercepts an arc equal to five times the length of the radius vector equals

1. 1 radian
2. 5 radians
3. 360 radians
4. 1/5 radian

2-9. Find the radian measure of a central angle in a circle with a radius of 3 inches, if the angle subtends an arc of 12 inches.

1. 36 radians
2. 1/4 radian
3. 90 radians
4. 4 radians

2-10. Express 220° in radians, using $1^\circ = \pi/180$ radians.

1. $4\pi/3$
2. $9\pi/11$
3. $11\pi/9$
4. $11\pi/8$

2-11. Express $16\pi/9$ in degrees, using 1 radian = $180^{\circ}/\pi$.

1. 32°
2. 320°
3. 360°
4. $320\pi^{\circ}$

2-12. If the length of the radius of a circle is 21 meters, then the length of arc subtended by a central angle of 7 radians is

1. 147 meters
2. 1/3 meter
3. 3 meters
4. 49/60 meter

2-13. The 10-inch long spokes on a wheel form an angle of 6° with each other at the axle. What is the length (in inches) of the arc between the outer ends of the spokes? (Hint: Convert degrees to radians.)

1. 60
2. $\pi/3$
3. $\pi/300$
4. 60π

2-14. If a printing cylinder 8 inches in diameter turns at 1,000 revolutions per minute, the number of feet of paper that travels across this cylinder in 1 hour is

1. 333 1/3
2. 4,000
3. 20,000
4. $40,000\pi$

● Items 2-15 and 2-16 are based on an automobile traveling at 80 miles per hour with wheel radius of 14 inches.

2-15. What is the angular velocity of the wheels in radians per second?

1. 5.71
2. 8.38
3. 100.57
4. 1,642.67

2-16. What is the angular velocity of the wheels in revolutions per minute? (Let π = 3.142.)

1. 10.53
2. 54.52
3. 80.01
4. 960.26

2-17. In a circle with a radius of 3 inches, a sector with a central angle of 45° has an area of how many square inches?

1. $3\pi/8$
2. $9\pi/8$
3. 9/8
4. 405/2

2-18. In a circle with a diameter of 12 inches, the area (in square inches) of a sector that has an arc length of $3\pi/2$ inches is

1. $2\pi/9$
2. $3\pi/4$
3. $9\pi/2$
4. 9π

2-19. The major practical application of mil measurement is

1. construction work
2. excavation projects
3. measurement of precious stones
4. ranging and sighting

2-20. How many degrees is equivalent to 144 mils?

1. 8.1
2. 81
3. 2,560
4. 6,400

2-21. An angle of 9 degrees is equal to how many mils?

1. 81/160
2. 160/81
3. 160
4. 360

2-22. Approximately how many mils make up 1 radian?

1. 0.001
2. 1/6400
3. 1,000
4. 6,400

2-23. An approximate radian measure for 360 mils is

1. 0.036
2. 0.36
3. 6,400
4. 360,000

2-24. What is an approximate measurement in mils for 7.3 radians?

1. 0.0073
2. 0.41
3. 129.78
4. 7,300

2-25. If the range of a tower is 1,000 feet and the tower subtends an angle of 100 mils, what is the height of the tower in feet?

1. 10
2. 100
3. 200
4. 1,000

2-26. If an enemy ship known to be 400 feet long and perpendicular to the line of sight subtends an angle of 20 mils, what is the ship's range in feet?

1. 50
2. 20,000
3. 24,000
4. 200,000

Learning Objective:

Apply properties of right triangles to problem solving.

2-27. Using the Pythagorean theorem, find the length (in inches) of the hypotenuse of a right triangle whose legs are 6 and 8 inches long.

1. 10
2. $\sqrt{14}$
3. $\sqrt{28}$
4. 100

2-28. If the two legs of an isosceles triangle are perpendicular to each other and one of the legs is 8 meters long, what is the length of the hypotenuse in meters?

1. 8
2. 16
3. $8\sqrt{2}$
4. 4

2-29. In a right triangle the shorter leg is 6 inches long and the hypotenuse is 12 inches long. The length of the longer leg (in inches) is

1. $\sqrt{6}$
2. $3\sqrt{2}$
3. $6\sqrt{5}$
4. $6\sqrt{3}$

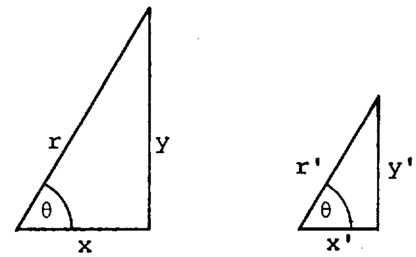


Figure 2A.--Similar triangles.

IN ANSWERING ITEMS 2-30 AND 2-31, REFER TO FIGURE 2A.

2-30. Since the two triangles are similar, which one of the following proportions is TRUE?

1. $\frac{r}{r'} = \frac{y}{y'}$
2. $\frac{x}{x'} = \frac{r'}{r}$
3. $\frac{y}{x} = \frac{x'}{y'}$
4. $\frac{y}{r} = \frac{r'}{y'}$

2-31. If r = 10, r' = 6, and x = 5, then the length of x' is

1. 8 1/3
2. 2
3. 3
4. 12

2-32. If one acute angle of a right triangle is 38°, then the other acute angle is

1. 38°
2. 52°
3. 62°
4. 90°

Learning Objective:

Apply trigonometric ratios, functions, and tables to problem solving.

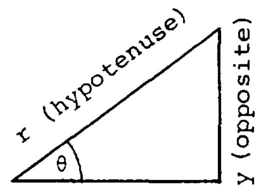


Figure 2B.--Right triangle.

IN ANSWERING ITEMS 2-33 THROUGH 2-38, REFER TO FIGURE 2B.

In items 2-33 through 2-35, select from column B the trigonometric function that corresponds to each ratio listed in column A.

A. RATIOS	B. FUNCTIONS
2-33. $\frac{y}{r}$	1. cos θ
	2. sin θ
2-34. $\frac{y}{x}$	3. tan θ
2-35. $\frac{r}{y}$	4. csc θ

In items 2-36 through 2-38, select from column B the ratio that corresponds to each trigonometric function in column A.

A. FUNCTIONS	B. RATIOS
2-36. cos θ	1. $\frac{\text{opposite}}{\text{hypotenuse}}$
2-37. sec θ	2. $\frac{\text{adjacent}}{\text{hypotenuse}}$
2-38. cot θ	3. $\frac{\text{adjacent}}{\text{opposite}}$
	4. $\frac{\text{hypotenuse}}{\text{adjacent}}$

2-39. Which of the following trigonometric functions, if any, are suggested if the hypotenuse and an acute angle of a right triangle are known and the side opposite the known acute angle is sought?

1. Tangent and cotangent
2. Sine and cosecant
3. Cosine and secant
4. None are suggested

2-40. Which of the following trigonometric functions, if any, are suggested when one leg and an acute angle of a right triangle are known and the length of the other leg is sought?

1. Cosine and secant
2. Sine and cosecant
3. Tangent and cotangent
4. None are suggested

2-41. In a right triangle, the side opposite θ is 6 and the side adjacent to θ is 12. The value of tan θ is

1. 1/2
2. 2
3. $\sqrt{5}$
4. $\sqrt{5}/2$

2-42. Find the value of csc θ in a right triangle if the hypotenuse is 17 and the side adjacent to θ is 15.

1. 15/17
2. 17/15
3. 8/17
4. 17/8

2-43. Trigonometric tables are lists of values of trigonometric functions.

1. True
2. False

● In answering items 2-44 through 2-46, refer to appendixes II and III.

2-44. The value of sin 30° is

1. 0.00873
2. 0.50000
3. 0.50754
4. 0.86603

2-45. The value of cos 48° 14' is

1. 0.75203
2. 0.74586
3. 0.66610
4. 0.65913

2-46. Find the value of tan 13° 36'.

1. 0.23823
2. 0.24193
3. 4.13350
4. 4.19756

In items 2-47 through 2-49, select from column B the trigonometric function that is equivalent to the expression in terms of sine and cosine listed in column A.

	A. EXPRESSIONS	B. FUNCTIONS
2-47.	$\frac{\sin \theta}{\cos \theta}$	1. sec θ
		2. cot θ
2-48.	$\frac{1}{\cos \theta}$	3. csc θ
2-49.	$\frac{\cos \theta}{\sin \theta}$	4. tan θ

2-50. If the hypotenuse of a right triangle is 12 feet long and one acute angle is 38°, what is the length (in feet) of the side opposite the given acute angle?

1. 7.38792
2. 9.37548
3. 9.45612
4. 19.4913

Assignment 3

Textbook assignment: Chapter 4, "Trigonometric Analysis," pages 4-1 through 4-35.

Learning Objective:

Apply trigonometric relationships to angles generated in the Cartesian coordinate system and recognize characteristics of the graphs of the sine, cosine, and tangent functions.

● In items 3-1 and 3-2, refer to the Cartesian coordinate system.

3-1. The number 5 is the y coordinate of the point (-7,5).

1. True
2. False

3-2. In what quadrant is the point (-3,-8) located?

1. I
2. II
3. III
4. IV

3-3. An angle is in standard position when its vertex is at the origin and its initial side is lying along the positive X axis of a Cartesian coordinate system.

1. True
2. False

3-4. The quadrant in which an angle lies is determined by the

1. length of the initial side
2. length of the terminal side
3. quadrant in which the initial side lies
4. quadrant in which the terminal side lies

3-5. Coterminal angles are two or more angles in standard position that have their terminal sides located at different positions.

1. True
2. False

3-6. Which of the following angles is coterminal to the general angle θ?

1. $\theta \pm 180^{\circ}$
2. $\theta \pm 360^{\circ}$
3. $\theta \pm 540^{\circ}$
4. $\theta \pm 630^{\circ}$

3-7. Which of the following sets of angles is coterminal?

1. -370°, 10°, 370°
2. 0°, 90°, 450°
3. -235°, 125°, 485°
4. -180°, 0°, 180°

In answering items 3-8 through 3-10, select from column B the ratio that corresponds to each of the trigonometric functions listed in column A.

	A. FUNCTIONS	B. RATIOS
3-8.	sin θ	1. $\frac{\text{length of radius}}{\text{abscissa}}$
3-9.	sec θ	2. $\frac{\text{ordinate}}{\text{abscissa}}$
3-10.	tan θ	3. $\frac{\text{length of radius}}{\text{ordinate}}$
		4. $\frac{\text{ordinate}}{\text{length of radius}}$

3-11. If a point on the radius vector has the coordinates (6,8), then the value of tan θ is

1. 4/5
2. 3/5
3. 3/4
4. 4/3

3-12. If P has the coordinates $(5,\sqrt{11})$, then sec θ is

1. 6/5
2. 5/6
3. $5\sqrt{11}/11$
4. $\sqrt{11}/5$

3-13. The signs of cosine and secant are positive in quadrants

1. I and II
2. I and III
3. I and IV
4. II and III

3-14. The signs of sine and cosecant are positive in quadrants

1. I and II
2. I and III
3. II and III
4. III and IV

3-15. The signs of tangent and cotangent are positive in quadrants

1. I and II
2. I and III
3. II and III
4. III and IV

3-16. If $\sin\theta = -3/5$, then $\cos\theta =$

1. either -5/4 or 5/4
2. either -3/4 or 3/4
3. either -3/5 or 3/5
4. either -4/5 or 4/5

3-17. In what quadrant does $\sin\theta = 1/2$ and $\tan\theta < 0$ occur?

1. I
2. II
3. III
4. IV

In items 3-18 through 3-20, select from column B the value of each function of θ listed in column A, if $\csc\theta = -2$ and $\cot\theta > 0$.

	A. FUNCTIONS	B. VALUES
3-18.	cos θ	1. $-2\sqrt{3}/3$
3-19.	tan θ	2. $-\sqrt{3}/2$
3-20.	sec θ	3. $-\sqrt{3}$
		4. $\sqrt{3}/3$

3-21. An angle of 220° has a reference angle of

1. -40°
2. 40°
3. 50°
4. 400°

3-22. A purpose of reduction formulas is to provide a means of expressing a function of any angle as a function of a positive acute angle.

1. True
2. False

3-23. If θ is a positive acute angle, then $(180^\circ + \theta)$ would be an angle in quadrant

1. I
2. II
3. III
4. IV

In answering items 3-24 through 3-29, select from column B the function of the angle less than 90° that is equivalent to the function of the angle greater than 90° listed in column A.

	A. FUNCTIONS OF ANGLES > 90°	B. FUNCTIONS OF ANGLES < 90°
3-24.	$\sin(180^\circ - \theta)$	1. sin θ
3-25.	$\sin(180^\circ + \theta)$	2. cos θ
3-26.	$\sin(360^\circ - \theta)$	3. -sin θ
3-27.	$\cos(180^\circ - \theta)$	4. -cos θ
3-28.	$\cos(180^\circ + \theta)$	
3-29.	$\cos(360^\circ - \theta)$	

3-30. Which of the following functions is equivalent to $\tan(180^\circ - \theta)$ and $\tan(360^\circ - \theta)$?

1. -tan θ
2. -cot θ
3. tan θ
4. cot θ

- Use the appropriate reduction formula in answering items 3-31 and 3-32.

3-31. The value of cot 250° is

1. 0.36397
2. -0.36397
3. -2.74748
4. 2.74748

3-32. The value of sin (-204°) is

1. -0.40674
2. 0.40674
3. 0.91355
4. -0.91355

3-33. The value of cos $1,087^\circ$ is

1. -0.12187
2. 0.12187
3. -0.99255
4. 0.99255

3-34. Which of the following functions is the cofunction of sec 48°?

1. sin 42°
2. sec 42°
3. csc 42°
4. cos 48°

3-35. In determining the values of the functions of 30-degree and 60-degree angles, we chose a radius vector of 2 units for convenience only. Had any other length been chosen, the ratios of the sides would have remained the same.

1. True
2. False

3-36. In a 30°-60°-90° triangle, the side opposite the 30° angle has a length equal to

1. $\sqrt{3}$
2. one-half the hypotenuse
3. the hypotenuse
4. 1, in all cases

In answering items 3-37 through 3-41, select from column B the value of the functions in column A, using geometric values of 30°-60° angles.

	A. FUNCTIONS	B. VALUES
3-37.	csc 60°	1. $\sqrt{3}$
3-38.	cot 30°	2. 1/2
3-39.	tan 240°	3. $\sqrt{3}/3$
3-40.	cos 300°	4. $2\sqrt{3}/3$
3-41.	sec 330°	

3-42. If the legs of a right isosceles triangle are one unit in length, then the length of the hypotenuse is

1. $\sqrt{2}$
2. $\sqrt{3}$
3. 2
4. $2\sqrt{2}$

In answering items 3-43 through 3-47, select from column B the value of the functions in column A, using geometric values of 45° angles.

	A. FUNCTIONS	B. VALUES
3-43.	sin 45°	1. 1
3-44.	sec 45°	2. $\sqrt{2}/2$
3-45.	tan 225°	3. -1
3-46.	cot 135°	4. $\sqrt{2}$
3-47.	csc 135°	

3-48. An angle in standard position whose terminal side lies on one of the coordinate axes is called which, if any, of the following angles?

1. Acute
2. Terminal
3. Quadrantal
4. None of the above

3-49. The value of any trigonometric function of a quadrantal angle whose ratio has a denominator of zero is zero.

1. True
2. False

In answering items 3-50 through 3-54, select from column B the value of the function of each quadrantal angle listed in column A.

	A. FUNCTIONS	B. VALUES
3-50.	tan 360°	1. 1
3-51.	sin 180°	2. -1
3-52.	sec 810°	3. 0
3-53.	tan 90°	4. undefined
3-54.	csc 270°	

3-55. Without using the appendixes, determine the value of $\cos^2 225^\circ + \sin^2 225^\circ$.

1. 1
2. $\sqrt{2}/2$
3. $-\sqrt{2}$
4. $\sqrt{2}$

3-56. The period of the sine function is

1. 90°
2. 180°
3. 270°
4. 360°

3-57. According to the graph of the sine function, as an angle increases from 0° to 90°, the value of the sine function of the angle will

1. decrease from positive one to zero
2. decrease from positive one to negative one
3. increase from zero to positive one
4. increase from negative one to positive one

3-58. The period of the cosine function is

1. $\pi/2$
2. π
3. $3\pi/2$
4. 2π

3-59. According to the graph of the cosine function, as an angle increases from 90° to 180°, the value of the cosine function of the angle will

1. decrease from zero to negative one
2. decrease from positive one to negative one
3. increase from negative one to zero
4. increase from negative one to positive one

3-60. The period of the tangent function is

1. 90°
2. 180°
3. 270°
4. 360°

3-61. The graph of the tangent function is discontinuous when the angle is

1. 0° or 180°
2. 0° or 360°
3. 90° or 270°
4. 180° or 360°

Assignment 4

Textbook assignment: Chapter 5, "Oblique Triangles," pages 5-1 through 5-23, Chapter 6, "Trigonometric Identities and Equations," pages 6-1 through 6-39, and Chapter 7, "Vectors and Forces," pages 7-1 through 7-27.

Learning Objective:

Apply the Law of Sines and the Law of Cosines in solving oblique triangles.

4-1. A triangle that contains no right angle is an oblique triangle.

1. True
2. False

4-2. Which of the following cases is NOT one of the four standard cases for solving oblique triangles?

1. One side and two angles
2. Two sides and the included angle
3. All three sides
4. All three angles

4-3. The statement, "the lengths of the sides of any triangle are proportional to the sines of their opposite angles," describes the Law of

1. Right Triangles
2. Tangents
3. Sines
4. Cosines

4-4. Which of the following equations is the correct form of the Law of Sines?

1. $x^2 + y^2 = r^2$
2. $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$
3. $\frac{b}{\sin A} = \frac{c}{\sin B} = \frac{a}{\sin C}$
4. $a^2 = b^2 + c^2 - 2bc \cos A$

4-5. Which of the following laws or theorems can best be used in solving an oblique triangle where two angles and a side are known?

1. Law of Tangents
2. Law of Cosines
3. Law of Sines
4. Pythagorean theorem

4-6. Use the values of the functions of special angles to find the length (in inches) of side a in triangle ABC if $A = 45^\circ$, $C = 75^\circ$, and side $b = 8$ inches.

1. $2\sqrt{6}/3$
2. $8\sqrt{2}$
3. $16\sqrt{3}/3$
4. $8\sqrt{6}/3$

4-7. In triangle ABC if $A = 35^\circ$, $C = 60^\circ$, and side $b = 12$, which of the following solutions is correct for triangle ABC?

1. $B = 85^\circ$, $a = 6$, $c = 9$
2. $B = 85^\circ$, $a = 6.9$, $c = 10.4$
3. $B = 78^\circ$, $a = 5$, $c = 11.2$
4. $B = 85^\circ$, $a = 8$, $c = 11.2$

4-8. In triangle ABC, angle A is acute. If you are given angle A, side a, and side c, it is possible to construct two triangles. What must be the relationship between side a and side c? (Note: Since side b is not given, it is the only side that may vary in length.)

1. $a < c$
2. $a \geq c$
3. $a > c$
4. $a = c$

4-9. If angle A is an obtuse angle and side a is equal to or less than side c, how many triangles, if any, may be constructed with these parts?

1. 1
2. 2
3. 3
4. None

4-10. How many solutions, if any, does triangle ABC have if angle $A = 20^{\circ}$, side a = 15 inches, and side c = 6 inches?

1. 1
2. 2
3. None
4. An infinite number

4-11. If in triangle ABC, angle $B = 30^{\circ}$, side b = 8 inches, and side c = 14 inches, then angle C is equal to

1. 88° 57' only
2. 31° 3' or 88° 57'
3. 61° 3' or 118° 57'
4. 61° 3' only

4-12. Which, if any, of the following solutions is correct concerning triangle ABC when $A = 60^{\circ}$, b = 10, and a = 5?

1. $B = 25^{\circ}$ 40', $C = 94^{\circ}$ 20', c = 11.5
2. $B = 35^{\circ}$ 16', $C = 84^{\circ}$ 44', c = 5.7
3. $B = 89^{\circ}$ 33', $C = 30^{\circ}$ 27', c = 8.7
4. No solution is possible

4-13. Which of the following equations is/are the correct form of the Law of Cosines?

1. $a^2 = b^2 + c^2 - 2bc \cos A$
2. $b^2 = a^2 + c^2 - 2ac \cos B$
3. $c^2 = a^2 + b^2 - 2ab \cos C$
4. All of the above

4-14. Which of the following laws or theorems can best be used in solving an oblique triangle where two sides and an angle between them are given or where three sides are given?

1. Law of Sines
2. Law of Tangents
3. Law of Cosines
4. Pythagorean theorem

4-15. If two sides of a triangle are 10 feet and 14 feet in length and the angle between these sides is 60°, what is the length, in feet, of the remaining side?

1. 7.3
2. 9.8
3. 12.5
4. 20.9

4-16. Which, if any, of the following angles is the solution to triangle ABC if sides a = 8, b = 10, and c = 12?

1. $A = 55^{\circ}$ 46', $B = 41^{\circ}$ 25', $C = 82^{\circ}$ 49'
2. $A = 41^{\circ}$ 25', $B = 55^{\circ}$ 46', $C = 82^{\circ}$ 49'
3. $A = 41^{\circ}$ 25', $B = 34^{\circ}$ 14', $C = 104^{\circ}$ 21'
4. No solution is possible

4-17. If the sides of a triangular slab measure 2 yards by 4 yards by 5 yards, what are the sizes of the angles opposite these sides?

1. 22° 20', 49° 27', 108° 13'
2. 22° 40', 49° 33', 107° 57'
3. 22° 20', 71° 2', 86° 38'
4. 18° 43', 79° 55', 81° 22'

Learning Objective:

Determine the area of an oblique triangle.

4-18. The formula area = $\frac{ab \sin C}{2}$ is used to find the area of a triangle when

1. the base and the altitude are given
2. two sides and the included angle are given
3. two sides and the angle opposite one of them are given
4. three sides are given

4-19. The area of a triangle, where $A = 30^\circ$, b = 8 inches, and c = 5 inches, is

1. 40 square inches
2. 20 square inches
3. 17.3 square inches
4. 10 square inches

4-20. The formula area = $\frac{c^2 \sin A \sin B}{2 \sin C}$, derived from the Law of Sines, is used to find the area of a triangle when

1. the base and the altitude are known
2. two sides and an angle are known
3. two angles and a side are known
4. three sides are known

In answering items 4-21 through 4-23, select from column B the appropriate formula for finding the area of a triangle PQR having the parts listed in column A.

A. PARTS	B. FORMULAS
4-21. p, q, R	1. area = $\frac{p^2 \sin Q \sin R}{2 \sin P}$
4-22. P, R, p	2. area = $\frac{pq \sin R}{2}$
4-23. q, Q, R	3. area = $\frac{q^2 \sin P \sin R}{2 \sin Q}$
	4. area = $\frac{qr \sin P}{2}$

4-24. The area of a triangle, where $A = 35^\circ$, $B = 50^\circ$, and b = 8 inches, is

1. 3.0 square inches
2. 14.1 square inches
3. 23.9 square inches
4. 42.6 square inches

Learning Objective:

Apply trigonometric identities and functions to problem solving.

4-25. An equivalent expression for sec x csc x cot x is

1. $\sin^2 x$
2. $\cos^2 x$
3. $\sec^2 x$
4. $\csc^2 x$

4-26. An equivalent expression for $\frac{\csc \theta}{\cot \theta + \tan \theta}$ is

1. $\sin \theta$
2. $\cos \theta$
3. $\csc \theta$
4. $\sec \theta$

4-27. The expression $(1 - \sin^2 x)(1 + \tan^2 x)$ is equivalent to

1. 1
2. $\sin^2 x$
3. $\tan^2 x$
4. 0

4-28. The expression $\sec(-\theta) - \tan(-\theta)\sin(-\theta)$ is equivalent to

1. $\cos \theta$
2. $-\cos \theta$
3. $\frac{-1 - \sin^2 \theta}{\cos \theta}$
4. $\frac{1 + \sin^2 \theta}{\cos \theta}$

4-29. Which of the following equations is an identity?

1. $\cos x (\sec x - \cos x) = \sin x$
2. $\cos x (\sec x - \cos x) = 1 - \cos x$
3. $\cos x (\sec x - \cos x) = \cot x - \cos^2 x$
4. $\cos x (\sec x - \cos x) = \sin^2 x$

4-30. The value of sec (135°) is

1. $\sqrt{2}$
2. $-\sqrt{2}$
3. $\sqrt{2}/2$
4. $-\sqrt{2}/2$

4-31. A simplified expression for $\tan(180^\circ - \beta)$ is

1. $\tan \beta$
2. $\dfrac{1 - \tan \beta}{1 + \tan \beta}$
3. $-\tan \beta$
4. $\dfrac{1 + \tan \beta}{1 - \tan \beta}$

4-32. If $\cos \alpha = 2/5$ and $\sin \beta = 1$, where α is in quadrant IV and β is on the positive Y axis, then $\csc(\alpha + \beta)$ is equal to

1. 2/5
2. -2/5
3. 5/2
4. -5/2

4-33. If $\theta = 150^\circ$, then $\sin 2\theta$ is equal to

1. $-\sqrt{3}/2$
2. $\sqrt{3}/4$
3. $-1/2$
4. $\sqrt{3}/2$

4-34. If $\cos \theta = -1/3$ and θ is in the third quadrant, then $\cos 2\theta$ is equal to

1. 7/9
2. -7/9
3. -1/3
4. -8/9

● The signs that precede the radicals in the half-angle formulas must be selected according to the quadrant in which $\theta/2$ lies. For example,

if $0^\circ \leq \theta \leq 90^\circ$, then $0^\circ \leq \theta/2 \leq 45^\circ$

if $90^\circ \leq \theta \leq 180^\circ$, then $45^\circ \leq \theta/2 \leq 90^\circ$

if $180^\circ \leq \theta \leq 270^\circ$, then $90^\circ \leq \theta/2 \leq 135^\circ$

if $270^\circ \leq \theta \leq 360^\circ$, then $135^\circ \leq \theta/2 \leq 180^\circ$

if $360^\circ \leq \theta \leq 450^\circ$, then $180^\circ \leq \theta/2 \leq 225^\circ$

and so on for all angles.

4-35. What is the value of $\tan \frac{\theta}{2}$, if $\theta = \frac{5\pi}{3}$?

1. 1/3
2. $-\sqrt{3}$
3. $\sqrt{3}/3$
4. $-\sqrt{3}/3$

4-36. What is the value of $\sin \frac{\theta}{2}$, when $\cot \theta = \frac{5}{12}$, $\csc \theta$ is negative, and $0^\circ < \theta < 360^\circ$?

1. 9/13
2. $-2\sqrt{13}/13$
3. $-3\sqrt{13}/13$
4. $3\sqrt{13}/13$

4-37. Which of the following equations is NOT a notation for the inverse of the tangent function $y = \tan x$?

1. $x = \tan y$
2. $y = 1/\tan x$
3. $y = \arctan x$
4. $y = \tan^{-1} x$

4-38. The inverse of a particular trigonometric function has many values; but if we restrict the range of the inverse relationship, the value of the trigonometric function is called the

1. principal value
2. limiting value
3. primary value
4. positive value

4-39. The value of Arcsin $\sqrt{3}/2$ in degrees is

1. 30°
2. 45°
3. 60°
4. 75°

4-40. The value of $\text{Cot}^{-1}(4.76595)$ in degrees is

1. 11°
2. $78^\circ\ 9'$
3. $11^\circ\ 51'$
4. $168^\circ\ 9'$

4-41. The equation

$$\sin\left[\text{Arccos}\left(-\frac{\sqrt{3}}{2}\right) + \text{Arctan}\left(\frac{3}{4}\right)\right]$$

is equivalent to

1. $\frac{3 - 4\sqrt{3}}{10}$
2. $\frac{2 - 3\sqrt{3}}{10}$
3. $\frac{4 + 3\sqrt{3}}{10}$
4. $\frac{4 - 3\sqrt{3}}{10}$

4-42. A trigonometric equation is an equality that is true for

1. all values of the variable
2. some values but not necessarily for all values of the variable
3. all positive values of the variable
4. all negative values of the variable

4-43. If $\sqrt{3}\cot\theta - 1 = 0$ for $0^\circ \leq \theta < 360^\circ$, then $\theta =$

1. 60° and 240°
2. 30° and 210°
3. 120° and 300°
4. 150° and 330°

Learning Objective:

Apply properties of vectors, forces, and equilibrium to problem solving.

4-44. A vector quantity is one that has

1. magnitude only
2. direction only
3. both magnitude and direction
4. neither magnitude nor direction

4-45. If two vectors are equal, then they are of the same length, are parallel, and point in opposite directions.

1. True
2. False

4-46. The resultant vector, $\vec{R}$, of the vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$ can be represented by which of the following equations?

1. $\vec{R} = \vec{A} + \vec{B} + \vec{C}$
2. $\vec{R} = \vec{B} + \vec{C} + \vec{A}$
3. $\vec{R} = \vec{C} + \vec{A} + \vec{B}$
4. All of the above

4-47. What are the magnitudes of the horizontal and vertical components of a vector having a magnitude of 130 miles, which makes an angle of 68° with the horizontal component vector?

1. 2,401 miles and 14,641 miles
2. 49 miles and 121 miles
3. 18 miles and 112 miles
4. 7 miles and 11 miles

4-48. An automobile travels due east on a level road for 60 km. It then turns due north at an intersection and travels 80 km before stopping. What is the resultant displacement of the car?

1. $2\sqrt{35}$ km
2. 140 km
3. 100 km
4. 10,000 km

4-49. A person walks 3 km north, 4 km west, 5 km south, and 6 km east. What is the magnitude and direction of the resultant vector?

1. $2\sqrt{2}$ km, 45° south of east
2. $2\sqrt{2}$ km, 45° north of east
3. 2 km, 45° south of east
4. 8 km, 45° north of west

4-50. A force of 10 newtons at 50° and another force of 15 newtons at 130° act on the same point. What is the magnitude and direction of the resultant force?

1. 4 newtons, 9° 28'
2. 19.5 newtons, 80° 32'
3. 19.5 newtons, 99° 28'
4. 27.2 newtons, 99° 28'

4-51. A man pushes with 140 newtons of force on the handle of a lawnmower. The angle between the handle and the ground is 55°. What are the magnitudes of the horizontal and vertical components of this force?

1. 9.0 newtons and 10.7 newtons
2. 80.3 newtons and 114.7 newtons
3. 98.0 newtons and 199.9 newtons
4. 170.9 newtons and 244.1 newtons

4-52. What two conditions are required for a body at rest to be in equilibrium?

1. The body must have translatory motion and rotary motion
2. The body must have neither translatory motion nor rotary motion
3. The body must have translatory motion but no rotary motion
4. The body must have rotary motion but no translatory motion

4-53. Three people push on a box. One pushes east with a force of 20 pounds, another pushes northeast with a force of 30 pounds, while the third pushes south with a force of 35 pounds. What would be the magnitude and direction of another person applying an equilibriant force?

1. 69.7 pounds, 53° 45' north of east
2. 69.7 pounds, 53° 45' south of west
3. 43.4 pounds, 18° 31' north of west
4. 43.4 pounds, 18° 31' south of east

4-54. To have no translatory motion, what must be TRUE, in all cases, for the sum of the magnitudes of the horizontal force components, F_x, and the sum of the magnitudes of the vertical force components, F_y?

1. $F_x = 0$ only
2. $F_y = 0$ only
3. $F_x = 0$ and $F_y = 0$
4. $F_x = F_y$

4-55. A 170-pound man sits at the center of a hammock that is tied between two trees 14 feet apart. The hammock sags 2 feet at the center. What is the tension in the ropes at each end of the hammock?

1. 327.4 pounds
2. 309.3 pounds
3. 176.8 pounds
4. 88.4 pounds

4-56. The torque will be considered negative if a force tends to produce a counterclockwise rotation about an axis.

1. True
2. False

4-57. A person trying to move a boulder pushes straight downward with a force of 700 newtons on one end of a crowbar that is 2 meters long. If the crowbar makes an angle of 35° with the horizontal, what is the torque produced by this force? (Hint: A sketch might be helpful.)

1. 200.8 newton·meters
2. 286.7 newton·meters
3. 803.0 newton·meters
4. 1,146.8 newton·meters

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